Kleine AG Algebraische Geometrie und Zahlentheorie

Hilbert Modular Surfaces

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In this meeting of the Kleine AG we study Hilbert modular surfaces. They parameterize abelian varieties with real multiplication: if A is a polarized abelian variety, F a number field and $\varrho : F \hookrightarrow \operatorname{End} A \otimes \mathbb{Q}$ an embedding of algebras with self-adjoint image (with respect to the polarization), then F must necessarily be totally real, i.e. all embeddings of F into the complex numbers factor over $\mathbb{R}$. Also the degree of F is at most the dimension of A. In the case of equality we say that A has *real multiplication* by F. With some auxiliary choices one can then study coarse moduli spaces of abelian varieties with real multiplication by F. Over the complex numbers they are given as $\mathbb{H}^g/\Gamma$, where g is the degree of F and Γ is an arithmetic subgroup of $\mathrm{SL}_2(F)$. Here $\mathrm{SL}_2(F)$ acts on $\mathbb{H}^g$ by the map $\mathrm{SL}_2(F) \longrightarrow \mathrm{SL}_2(\mathbb{R})^g$ induced by the g embeddings $F \hookrightarrow \mathbb{R}$. Such moduli spaces are called *Hilbert modular varieties.*

The case $F = \mathbb{Q}$ gives precisely elliptic curves and their moduli spaces, and the natural first step to generalize their theory is then to consider the case where F is a real quadratic field. The obtained surfaces are then called *Hilbert modular surfaces* or *Hilbert-Blumenthal surfaces*, named after Hilbert's student Otto Blumenthal. Hilbert hoped that the study of these surfaces would bring progress to higher-dimensional complex analysis and algebraic geometry. In fact Hilbert modular surfaces were systematically studied only after the second world war, due to the mass of new techniques. Friedrich Hirzebruch wrote a whole series of papers on them, concentrating mainly on their geometric and topological properties, and the resolution of their singularities. But also their arithmetic properties were much studied; a major result was the proof of the Tate conjecture for Hilbert modular surfaces over abelian extensions of $\mathbb{Q}$ by Harder, Langlands and Rapoport [3]. There is also a great amount of literature on Hilbert modular forms, the natural generalization of elliptic modular forms.

The talks in this Kleine AG are relatively independent of each other, with the following two exceptions: all talks depend on the first one, and the fifth talk depends (at least in motivation) on the fourth one. For this reason we have also kept the summaries of the talks quite short, so that the speakers have some freedom to decide what to talk about.

First Talk: Introduction (45 minutes)**.** In this talk the speaker should introduce Hilbert modular surfaces as moduli varieties for abelian varieties with real multiplication. Thus we first need some generalities on real multiplication which are all not difficult but should be kept in mind. For the general definition of real multiplication see e.g. [2, sections 2.1—2.2] and [7, section 1]. The following should be mentioned: since $\mathscr{O}_F$ is not necessarily a principal ideal domain, there can be more than one isomorphism class of lattices with symplectic form and self-adjoint $\mathscr{O}_F$-operation. Every abelian variety with real multiplication by $\mathscr{O}_F$ can be polarized in way compatible with the real multiplication structure, see [7, Proposition 1.10]. Finally, a complex torus with real multiplication is automatically polarizable and hence algebraic.[3]

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[3] This is apparently not in any of the references given below, but the proof is very simple. Let V/L be a two-dimensional complex torus and let let $\varrho : F \longrightarrow \operatorname{End}(V/L) \otimes \mathbb{Q}$ be an embedding. Then $L \otimes F$ decomposes into two two-dimensional

For the rest of this talk and the entire next talk, we work over the complex numbers. Show that coarse moduli spaces for abelian varieties with real multiplication by $\mathscr{O}_F$ over $\mathbb{C}$ are given as quotients $\mathbb{H}^2/\operatorname{SL}(\mathfrak{a} \oplus \mathscr{O}_F)$ as explained in [1, chapter I] and [2, section 2.2]. The most important example is of course $\operatorname{SL}(\mathscr{O}_F^2) = \operatorname{SL}_2(\mathscr{O}_F)$. Show that $\mathbb{H}^2/\operatorname{SL}_2(\mathscr{O}_F)$ can be compactified by adding finitely many points, the *cusps*, which are in bijection with the ideal class group of F.

Second Talk: Resolution of the Cusp Singularities (45 minutes)**.** There can be singularities at the cusps, and these can be resolved in a particularly pretty way involving continued fractions. We advise the speaker to follow Friedrich Hirzebruch's Bourbaki talk [4]. If time is too short, §5 (Applications) may be skipped. For further details she might also consult [1, chapter V].

Third Talk: Hilbert Modular Surfaces as Shimura Varieties (30 minutes)**.** Since they parameterize abelian varieties with polarization and endomorphism structure, Hilbert modular surfaces are Shimura varieties. Hence they admit an adèlic description. The algebraic group in question is the Weil restriction of scalars $G = \operatorname{Res}_{F|\mathbb{Q}} \operatorname{GL}_2$. The speaker should be familiar with the adèlic description of Shimura varieties, as described in detail for example in Milne's online course notes. The content to be covered is roughly [1, section I.7]. It would be nice to see the identification of the complex points of the modular variety with the double coset space quite explicitly. In particular Corollary 7.3 should be mentioned and explained: the Shimura variety viewpoint naturally makes us consider all $\mathbb{H}^2/\hat{\Gamma}(\mathfrak{a} \oplus \mathscr{O}_F)$ instead of just $\mathbb{H}^2/\operatorname{SL}_2(\mathscr{O}_F)$. However, every component is defined over $\mathbb{Q}$ (this is stated, but not proved, in [1, X.4] ...)

Fourth Talk: Hirzebruch-Zagier Cycles (45 minutes)**.** In this and the next talk, we only consider the "standard" Hilbert modular surfaces $X = \mathbb{H}^2/\operatorname{SL}_2(\mathscr{O}_F)$. The *Hirzebruch-Zagier cycles* are algebraic cycles T_N on X introduced by Hirzebruch to determine the Kodaira dimension of X. We have a look instead at their intersection numbers. Again there is a quite nice article by Hirzebruch which can serve as an introduction, namely [5]. The point for us is the "Vermutung" on the turn of pages 91/92, i.e. the one with equation (24). It says that the intersection numbers of the T_N with a "nice enough" cycle on X form the coefficients of an elliptic modular form. The proof of this conjecture will be sketched in the final talk, so the major goal of this talk should be to introduce and explain the notions and objects involved in its statement.

The speaker may also consider [3] where Hirzebruch-Zagier cycles are described in the language of Shimura varieties.

Fifth Talk: Modular Forms of Nebentypus (60 minutes)**.** Here we follow the article [6] by Hirzebruch and Zagier, in wich the above mentioned conjecture was finally proved. The content to be covered is section 3.1, the conjecture has become Theorem 1 in that section. Sometimes you will need to refer to earlier sections of the article.

References

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subspaces V_1 and V_2 such that $a \in F$ operates by multiplication with a on V_1 and by multiplication with a^σ on V_2. Here σ is the nontrivial automorphism of F. In particular these two are interchanged by the Galois group of $F|\mathbb{Q}$. For dimension reasons one can find a unique Riemann form (up to scalar multiplication) on V_1. Take its conjugate on V_2, then their sum on $V \otimes F$ is defined over $\mathbb{Q}$ and hence gives a polarization

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