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# On Regularity for plasticity with hardening

Dominique Löbach

**Abstract**

In this paper we show the regularity of the strain tensor and local differentiability of the stress tensor and hardening parameters in plasticity with hardening using a viscoplastic type penalisation in the case of von Mises yield criterion. The regularity of the strain tensor was first shown by Johnson [Joh78] by constructing a bijection between the strain- and stress tensor. The local differentiability was shown by Seregin [Ser94] with a dual method. In this paper we can bound the strain tensor of the penalized problem on the unit sphere in $L^2(L^2)$ and obtain uniform results in the passage to the limit. This is shown in a more direct way.

**Keywords** : plasticity with hardening, kinematic hardening, isotropic hardening regularity of solutions

**Subject classification(2000)** : primary 74C05 secondary: 35B60, 35K85

## Contents

# 1 The hardening problem

Let $\Omega \subset \mathbb{R}^n$ be an open bounded and connected subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and $\partial\Omega = \Gamma_N \dot{\cup} \Gamma_D$.

We consider the functions

$$\begin{aligned}
&\sigma : \Omega \times [0,T] \to \mathbb{R}^{n\times n}_{\text{sym}} \\
&\xi : \Omega \times [0,T] \to \mathbb{R}^m \\
&u : \Omega \times [0,T] \to \mathbb{R}^n \,.
\end{aligned}$$

Where $\sigma$ represents the stress tensor, $\xi$ the internal hardening parameters and $u$ the displacement field. The variable $t$ has the character of a loading parameter.

Let $A \in L^\infty\big(\Omega; \hom(\mathbb{R}^{n\times n}_{\text{sym}}, \mathbb{R}^{n\times n}_{\text{sym}})\big)$ be an uniformly elliptic, symmetric fourth order tensor field with ellipticity constant $\alpha_A > 0$. That is:

$$\big(A(x)m\big) : m \geq \alpha_A |m|^2 \quad \forall\, m \in \mathbb{R}^{n\times n}_{\text{sym}} \,.$$

This tensor describes the elastic material properties and is an inverse Hookean law (for example the inverse Lamé-Navier Operator).
Further let the hardening modulus $H \in L^\infty(\Omega; \mathbb{R}^{m\times m})$ be an symmetric, uniform elliptic second order tensor field with ellipticity constant $\alpha_H > 0$.

We define the linearized strain tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\intercal)$.

Let $\mathcal{F} : \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^m \to \mathbb{R}$ be a continuous convex function. We call $\mathcal{F}$ a yield function. The yield function $\mathcal{F}$ models the behaviour of the material, that means decides if the material is in an pure elastic respective plastic state.

We assume that $\overline{\Omega}$ is subjected to the following body- and surface force densities

$$\begin{aligned}
&f \in L^\infty(0,T; L^n(\Omega, \mathbb{R}^n)) \\
&p \in L^\infty(0,T; L^\infty(\partial\Omega, \mathbb{R}^n)) \,.
\end{aligned} \tag{1.1}$$

We consider the following sets of admissible stresses and hardening parameters :

$$\mathcal{K}=\big\{(\tau,\eta)\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))\times L^2(0,T;L^2(\Omega,\mathbb{R}^m))\,|\,\mathcal{F}(\tau,\eta)\leq 0 \text{ a.e. in } \Omega\times[0,T]\big\}$$

$$\mathcal{M}=\big\{(\tau,\eta)\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))\times L^2(0,T;L^2(\Omega,\mathbb{R}^m))|\tau\cdot\vec{n}=p \text{ on } \Gamma_N, \operatorname{div}\tau\in L^\infty(0,T;L^n(\Omega,\mathbb{R}^n))\big\}$$

Let $BD(\Omega)$ denote the space of functions with bounded deformation, that is

$$BD(\Omega)=\big\{u\in L^1(\Omega,\mathbb{R}^n)\,|\,\varepsilon(u)\in\big(C_o(\Omega,\mathbb{R}^n)\big)^*\big\}.$$

This means the strain tensor $\varepsilon(u)$ is only a bounded measure. The space $BD(\Omega)$ can be continuously embedded into $L^{\frac{n}{n-1}}(\Omega,\mathbb{R}^n)$. For further information about $BD(\Omega)$ see Temam [Tem85].

We write $v=\frac{\partial}{\partial t}u(x,t)$ for the "displacement velocity".

**Definition 1.1** *The variational inequality of plasticity with hardening is to find* $\big((\sigma,\xi),v\big)\in(\mathcal{M}\cap\mathcal{K})\times L^1(0,T;BD(\Omega))$ *such that* $\dot\sigma,\dot\xi\in L^2(L^2)$ *and for all* $(\tau,\eta)\in\mathcal{K}\cap\mathcal{M}$ *holds*

$$\big(A\dot\sigma,\tau-\sigma\big)+\big(H\dot\xi,\eta-\xi\big)+\langle v,\operatorname{div}(\tau-\sigma)\rangle\geq 0 \tag{1.2}$$

$$\begin{aligned}\langle\sigma,\nabla w\rangle&=\langle f,w\rangle+\int_{\Gamma_N}pw\,\mathrm{d}\Gamma\quad\forall\, w\in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n)\ \textit{a.e. with respect to } t\\(\sigma,\xi)(0)&=0\ \textit{in}\ \Omega\times\mathbb{R}^m\times\{t=0\}\\v&=0\ \textit{on}\ \Gamma_D\times[0,T]\,.\end{aligned}\tag{1.3}$$

In this paper we investigate the case of (linear) isotropic and kinematic hardening in the case of von Mises yield criterion.

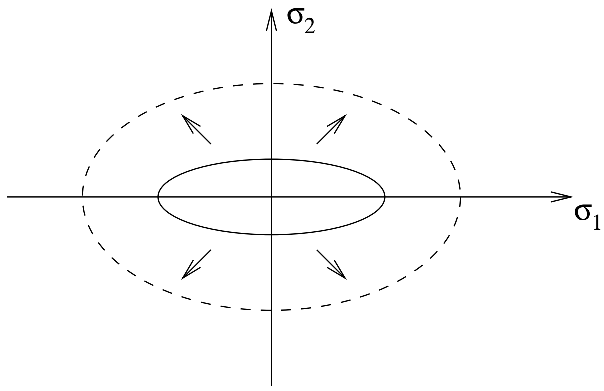
$$\mathcal{F}(\sigma,\xi)=|\sigma_D|-(\kappa+\xi)\qquad\text{isotropic hardening, } \xi\in\mathbb{R}$$

$$\mathcal{F}(\sigma,\xi)=|\sigma_D-\xi_D|-\kappa\qquad\text{kinematic hardening, } \xi\in\mathbb{R}^{n\times n}_{\text{sym}}$$

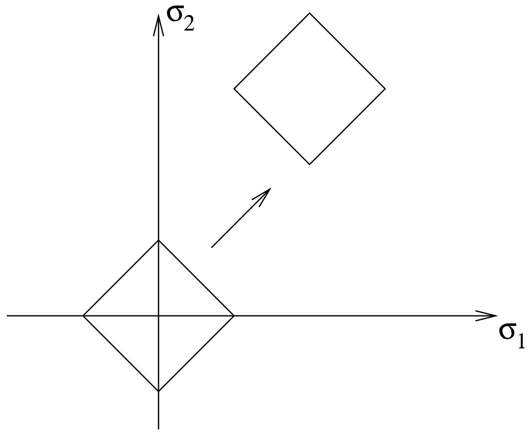
with $\kappa>0$ the yield limit and $\sigma_D=\sigma-\frac{1}{n}\operatorname{tr}(\sigma)Id$ the deviator of $\sigma$.

The isotropic hardening describes an uniform expansion of the yield surface, and the yield function takes the form: $\mathcal{F}(\sigma,\xi)=F(\sigma)-\kappa(\xi)$, where $F$ is a continuous, convex and $\kappa$ a continuous, concave function.

By kinematic hardening the yield surface undergoes a translation but no change of size or it's shape. The yield function can be written as $\mathcal{F}(\sigma, \xi) = F(\sigma - \xi) - \kappa$ with $F$ continuous and convex.




(a) Isotropic




(b) Kinematic

Figure 1: Isotropic and Kinematic hardening

## 2 The penalized hardening problem

We will use a viscoplastic type approximation to the hardening problem (1.2). Let us assume, that our yield function $\mathcal{F}(\sigma, \xi)$ is (weak) differentiable. Let $\mu > 0$, we define our viscoplastic potential $G_\mu(\sigma, \xi)$ as

$$G_\mu(\sigma, \xi) := \frac{1}{2\mu}\big(\mathcal{F}(\sigma, \xi)\big)_+^2 \tag{2.1}$$

where $(a)_+ = \begin{cases} a & \text{if } \ a \geq 0 \\ 0 & \text{if } \ a < 0\,. \end{cases}$

The term $G_\mu(\sigma, \xi)$ is almost everywhere (weak) differentiable and convex. Thus the derivative $G'_\mu(\sigma, \xi)$ is a monotone operator.
We further impose a linear growth condition on the viscoplastic potential derived from the yield function $\mathcal{F}$. Let $\mathcal{F}$, such that

$$|\nabla_{(\sigma,\xi)}\big(\mathcal{F}(\sigma, \xi)\big)_+^2| \leq c_1|\sigma| + c_2|\xi| + Const\,. \tag{2.2}$$

We now formulate the penalized problem of hardening.

**Definition 2.1** *Find* $\big((\sigma_\mu, \xi_\mu), v_\mu\big) \in \mathcal{M} \times L^1(0, T; BD(\Omega))$ *such that* $(\dot\sigma_\mu, \dot\xi_\mu) \in L^2(L^2)$ *and for all* $(\tau, \eta) \in \mathcal{M}$ *holds:*

$$\big(A\dot\sigma_\mu,\tau - \sigma_\mu\big) + \big(H\dot\xi_\mu,\eta - \xi_\mu\big) + \big(G'_\mu((\sigma_\mu, \xi_\mu)),(\tau - \sigma_\mu, \eta - \xi_\mu)\big) + \langle v_\mu, \operatorname{div}(\tau - \sigma_\mu)\rangle = 0 \tag{2.3}$$

$$\begin{aligned} \langle \sigma_\mu, \nabla w\rangle &= \langle f, w\rangle + \int_{\Gamma_N} pw\mathrm{d}\Gamma \quad \forall\, w \in H^1_{\Gamma_D}(\Omega, \mathbb{R}^n) \ \textit{a.e. with respect to } t \\ (\sigma_\mu, \xi_\mu)(0) &= 0 \ \textit{in } \Omega \times \mathbb{R}^m \times \{t = 0\} \\ v_\mu &= 0 \ \textit{on } \Gamma_D \times [0, T] \end{aligned} \tag{2.4}$$

The existence of solutions $(\sigma_\mu, \xi_\mu)$ will be shown in section 4 .
For $\mu \to 0$ the sequence $\big((\sigma_\mu, \xi_\mu), v_\mu\big)$ converges to the solution of the original hardening problem (1.2) . (For the proof see section 6 . )
In this paper we consider the von Mises yield criterion and we have:

$$G_\mu(\sigma_\mu, \xi_\mu) = \frac{1}{2\mu}\big(|\sigma_{\mu D}| - (\kappa + \xi_\mu)\big)_+^2 \qquad \text{isotropic hardening}$$

$$G_\mu(\sigma_\mu, \xi_\mu) = \frac{1}{2\mu}\big(|\sigma_{\mu D} - \xi_{\mu D}| - \kappa\big)_+^2 \qquad \text{kinematic hardening}$$

Our viscoplastic potential (2.1) extends the idea of an associated flow rule. In the context of the Prandtl-Reuss flow rule of perfect plasticity

$$\varepsilon(\dot{u}) = A\dot{\sigma} + \dot{\Pi} \tag{2.5}$$

where $\Pi$ denotes the plastic strain and $\dot{\Pi} = 0$ if $F(\sigma) \leq 0$ with $F : \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}$ a continuous convex yield function.

> **Attention:** The presentation of perfect plasticity in the form (2.5) is common among engenieers but only formal in the strict mathematical sense. In perfect plasticity the (elastic) strain tensor $\varepsilon(\dot{u})$ is in the worst case only a bounded measure, thus $\dot{u} \in BD(\Omega)$.
> Using formal Green's theorem for $\varepsilon$ and by the principle of maximum plastic dissipation (2.7), we derive the variational inequality
>
> $$\big(A\dot{\sigma},\tau - \sigma\big) + \langle \dot{u}, \operatorname{div}(\tau - \sigma)\rangle \geq 0\,. \tag{2.6}$$
>
> This inequality lacks the boundary and initial conditions, but is the basis for the mathematical formulation of perfect plasticity. (For more details about perfect plasticity see Suquet [Suq81].)

The engenieers introduce a plastic potential $\Phi$ such that

$$\dot{\Pi} = \dot{\lambda}\frac{\partial}{\partial\sigma}\Phi(\sigma)\,.$$

$\dot{\lambda}$ is a nonnegative (infinitesimal) scalar.
The plastic strain $\dot{\Pi}$ satisfies the the principle of maximum plastic dissipation (or maximum plastic work [DL76, HR99])

$$\dot{\Pi} : (\tau - \sigma) \leq 0 \qquad \forall\,\tau :\, \mathcal{F}(\tau) \leq 0\,. \tag{2.7}$$

In the context of plasticity with hardening we have ([KL84])

$$\dot{\Pi} = \dot{\lambda}\frac{\partial}{\partial\sigma}\Phi(\sigma,\xi)\,. \tag{2.8}$$

The flow rule (2.8) is called asssociated if $\Phi$ is given by the yield function $\mathcal{F}$

$$\dot{\Pi} = \dot{\lambda}\frac{\partial}{\partial\sigma}\mathcal{F}(\sigma,\xi)\,.$$

In the case of von Mises yield criterion (which we only consider in this paper) a potential of the form

$$\Psi = \frac{1}{2\mu}\big(\mathcal{F}\big)_+^2$$

is called associated to the Hohenemser-Prager model (see Lubliner [Lub90]). In the framework of Han and Reddy [HR99], we consider generalized stresses $\Sigma := (\sigma, \xi)$ and plastic strains $\Pi = (\pi_\sigma, \pi_\xi)$, the associated flow rule

$$\dot{\Pi} = \dot{\lambda}\nabla\mathcal{F}(\Sigma)$$

reads componentwise

$$\begin{aligned} \dot{\pi}_\sigma &= \dot{\lambda}\frac{\partial}{\partial\sigma}\mathcal{F}(\sigma,\xi) \\ \dot{\pi}_\xi &= \dot{\lambda}\frac{\partial}{\partial\xi}\mathcal{F}(\sigma,\xi)\,. \end{aligned} \tag{2.9}$$

In this context the derivative $G'_\mu$ of our viscoplastic potential (2.1) satisfies the extended principle of maximum plastic dissipation

$$\dot{\Pi} : (T - \Sigma) \leq 0 \qquad \forall\, T :\ \mathcal{F}(T) \leq 0\,.$$

The monotonicity of $G'_\mu$ gives for $(\tau, \eta) \in \mathcal{M} \cap \mathcal{K}$

$$\big(G'_\mu(\sigma,\xi), (\tau - \sigma, \eta - \xi)\big) \leq 0\,.$$

The choice (2.1) is an extension of the potential associated to the Hohenemser-Prager model.

## 3 A priori estimates for the penalized problem

If we make the assumption of a safe load condition (see Johnson [Joh76, Joh78]) one can obtain the existence and estimates for the solutions of the penalized problem independent of $\mu$.

**safe load condition:**

There exists a $(\tau,\eta) \in W^{1,\infty}(0,T;L^\infty(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}})) \times W^{1,\infty}(0,T;L^\infty(\Omega,\mathbb{R}^m))$ and $\delta > 0$ such that

$$\left.\begin{aligned} -\operatorname{div}\tau &= f \text{ in } \Omega\times[0,T] \\ \tau\cdot\vec{n} &= p \text{ on } \Gamma_N\times[0,T] \\ (\tau,\eta)(0) &= 0 \text{ in } \Omega\times\mathbb{R}^m\times\{t=0\} \\ F(\tau,\eta) &\le -\delta < 0 \end{aligned}\right\} \tag{3.1}$$

We can now give the following estimates for the sequence of solutions $(\sigma_\mu,\xi_\mu)$ of the penalized hardening model. The solvability is shown in section 4.

**Theorem 3.1** *Under the assumptions of section 1 and the safe load condition* (3.1)*, the solutions* $(\sigma_\mu,\xi_\mu)$ *of the penalized hardening model satisfy the following estimates independent of* $\mu$

$$\begin{aligned} \|\sigma_\mu\|_{L^\infty(L^2)} &\le Const \\ \|\xi_\mu\|_{L^\infty(L^2)} &\le Const\,. \end{aligned} \tag{3.2}$$

**proof** We choose in equation (2.3) $(\sigma_\mu-\tau,\xi_\mu-\eta)$ where $(\tau,\eta)$ satisfies the safe load condition (3.1)

$$\big(A\dot\sigma_\mu,\sigma_\mu-\tau\big)+\big(H\dot\xi_\mu,\xi_\mu-\eta\big)+\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big)=0\,.$$

Sorting terms

$$\big(A\dot\sigma_\mu,\sigma_\mu\big)+\big(H\dot\xi_\mu,\xi_\mu\big)+\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big)=\big(A\dot\sigma_\mu,\tau\big)+\big(H\dot\xi_\mu,\eta\big)\,. \tag{3.3}$$

Write $\big(A\dot\sigma_\mu,\sigma_\mu\big),\big(H\dot\xi_\mu,\xi_\mu\big)$ as time derivative

$$\begin{aligned} \big(A\dot\sigma_\mu,\sigma_\mu\big) &= \frac{1}{2}\frac{d}{dt}\big(A\sigma_\mu,\sigma_\mu\big) \\ \big(H\dot\xi_\mu,\xi_\mu\big) &= \frac{1}{2}\frac{d}{dt}\big(H\xi_\mu,\xi_\mu\big) \end{aligned}$$

and integrate (3.3) from 0 to $t$

$$\frac{1}{2}\big(A\sigma_\mu,\sigma_\mu\big)+\frac{1}{2}\big(H\xi_\mu,\xi_\mu\big)+\int_0^t\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big)\mathrm{d}s \le \int_0^t\big(A\dot\sigma_\mu,\tau\big)+\big(H\dot\xi_\mu,\eta\big)\mathrm{d}s\,. \quad (3.4)$$

The tested penalty term is definite, using the inequality for convex differentiable functions we obtain

$$\int_0^t \big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big)\mathrm{d}s \ge \int_0^t \underbrace{G_\mu((\sigma_\mu,\xi_\mu))}_{\ge 0} - \underbrace{G_\mu((\tau,\eta))}_{=0}\,\mathrm{d}s\,. \quad (3.5)$$

We use partial integration on the terms of the right hand side of (3.4). (note that $(\sigma_\mu,\xi_\mu)(0)=0$ and $A,H$ are symmetric)

$$\int_0^t \big(A\dot\sigma_\mu,\tau\big)+\big(H\dot\xi_\mu,\eta\big)\mathrm{d}s = \big(\sigma_\mu,A\tau\big)+\big(\xi_\mu,H\eta\big)-\int_0^t\big(\sigma_\mu,A\dot\tau\big)\mathrm{d}s-\int_0^t\big(\xi_\mu,H\dot\eta\big)\mathrm{d}s \quad (3.6)$$

Using Young's inequality gives

$$\begin{aligned}&\big(\sigma_\mu,A\tau\big)-\int_0^t\big(\sigma_\mu,A\dot\tau\big)\mathrm{d}s+\big(\xi_\mu,H\eta\big)-\int_0^t\big(\xi_\mu,H\dot\eta\big)\mathrm{d}s\\ &\le \gamma\|\sigma_\mu\|^2+\frac{1}{4\gamma}\|A\tau\|^2+\int_0^t\|\sigma_\mu\|^2\mathrm{d}s+\int_0^t\|A\dot\tau\|^2\mathrm{d}s\\ &\qquad +\rho\|\xi_\mu\|^2+\frac{1}{4\rho}\|H\eta\|^2+\int_0^t\|\xi_\mu\|^2\mathrm{d}s+\int_0^t\|H\dot\eta\|^2\mathrm{d}s\,. \end{aligned}\quad (3.7)$$

We choose $\gamma,\rho$ such that we can absorb terms, using the ellipticity of $A$ and $H$, equations (3.4) and (3.7) yield

$$(\frac{\alpha_A}{2}-\gamma)\|\sigma_\mu\|^2+(\frac{\alpha_H}{2}-\rho)\|\xi_\mu\|^2 \le Const+\int_0^t\|\sigma_\mu\|^2\mathrm{d}s+\int_0^t\|\xi_\mu\|^2\mathrm{d}s\,. \quad (3.8)$$

The Gronwall lemma implies that $\|\sigma_\mu\|$ and $\|\xi_\mu\|$ are bounded.
This proofs the statement of the theorem. □

These results lead to

**Theorem 3.2** *Under the assumptions of theorem 3.1, we have*

$$\begin{aligned}&\|G_\mu(\sigma_\mu,\xi_\mu)\|_{L^1(L^1)}\le Const\\ &\|G'_\mu(\sigma_\mu,\xi_\mu)\|_{L^1(L^1)}\le Const\,.\end{aligned}\quad (3.9)$$

**proof** The results from theorem 3.1 and lemma 2 from [Suq81] yield the estimates. □

# 4 Existence of solutions of the penalized hardening problem

We will now show the existence of solutions $\big((\sigma_\mu, \xi), v_\mu\big)$ and time derivatives of the stress tensor and hardening parameters in the penalized hardening model.

**Theorem 4.1** *Under the assumptions of section 1 and the safe load condition* (3.1)*, there exists a solution* $\big((\sigma_\mu, \xi_\mu), v_\mu\big)$ *of the penalized hardening problem* (2.3) *. The time derivatives of the stress tensor* $\dot\sigma_\mu$ *and the hardening parameter* $\dot\xi_\mu$ *exist and we have the estimates independent of the penalty parameter* $\mu$

$$\begin{aligned} &\|\dot\sigma_\mu\|_{L^2(L^2)} \le Const \\ &\|\dot\xi_\mu\|_{L^2(L^2)} \le Const\,. \end{aligned} \tag{4.1}$$

**proof**

**1)** We discretize the weak formulation (2.3) of penalized plasticity with hardening in time with finite backward differences.

Let $N \in \mathbb{N}^+$, $k = \frac{T}{N}$ the step size in time direction and $\eta^m = \eta(m\cdot k)$. Write

$$D_t^{-k}\eta^m = \frac{\eta^m - \eta^{m-1}}{k}$$

for finite backward differences in time. The time discretized formulation is now

$$\big(AD_t^{-k}\sigma_\mu^m,\sigma_\mu^m - \chi^m\big) + \big(HD_t^{-k}\xi_\mu^m,\xi_\mu^m - \omega^m\big) + \big(G'_\mu((\sigma_\mu^m, \xi_\mu^m)),(\sigma_\mu^m - \chi^m, \xi_\mu^m - \omega^m)\big) \\ + \langle v_\mu^m, \operatorname{div}(\sigma_\mu^m - \chi^m)\rangle = 0\,. \tag{4.2}$$

The balance of forces (1.3) is altered in the following way:

$$\begin{aligned} \operatorname{div}\sigma_\mu^m &= \frac{1}{k}\int_{m\cdot k}^{(m+1)\cdot k} f\mathrm{d}s \\ \sigma_\mu^m\cdot\vec n &= \frac{1}{k}\int_{m\cdot k}^{(m+1)\cdot k} p\mathrm{d}s \text{ on } \Gamma_N\,. \end{aligned} \tag{4.3}$$

We approximate the pair $(\tau, \eta)$ in the safe load condition(3.1) by

$$\begin{aligned} \tau^m &= \frac{1}{k}\int_{m\cdot k}^{(m+1)\cdot k} \tau\mathrm{d}s \\ \eta^m &= \frac{1}{k}\int_{m\cdot k}^{(m+1)\cdot k} \eta\mathrm{d}s\,, \end{aligned} \tag{4.4}$$

and as in (4.3) we let $(\tau^m, \eta^m)$ satisfy the altered balance of forces. By the convexity of $\mathcal{K}$ the altered a discrete analog of the safe load condition still holds

$$\mathcal{F}(\tau^m, \eta^m) < 0 \text{ a.e. in} \Omega. \tag{4.5}$$

The existence of a solution $(\sigma_\mu^m, \xi_\mu^m)$ on every time step $m$, of the discretized formulation (4.2) for $\mu, k$ fixed can be shown by direct methods in the calculus of variations.
We consider the energy functional $J_\mu^m$:

$$J_\mu^m(\sigma_\mu^m, \xi_\mu^m) := \frac{1}{2k}\big(A\sigma_\mu^m, \sigma_\mu^m\big) - \frac{1}{k}\big(A\sigma_\mu^m, \sigma_\mu^{m-1}\big) + \frac{1}{2k}\big(H\xi_\mu^m, \xi_\mu^m\big) - \frac{1}{k}\big(H\xi_\mu^m, \xi_\mu^{m-1}\big) + G_\mu(\sigma_\mu^m, \xi_\mu^m)\,. \tag{4.6}$$

The Euler-Lagrange equation of (4.6) is just the time discretized equation (4.2) without the term $\langle v_\mu^m, \operatorname{div}(\sigma_\mu^m - \chi^m)\rangle$.
The existence of the displacement velocity $v_\mu^m \in H^1_{\Gamma_D}(\Omega, \mathbb{R}^n)$ can be shown in a similar way to Anzelloti & Giaquinata [AG80, AG82] since $J_\mu^m$ is a perturbed Hencky like energy functional.

**2)** We now introduce the operator $I_p^k$ of piece wise constant interpolation in time as

$$I_p^k\sigma(t) := \frac{1}{k}\int_{m\cdot k}^{(m+1)\cdot k} \sigma^m(s)\mathrm{d}s\,, \quad m\cdot k \le t \le (m+1)\cdot k, \quad m = 1, \ldots, N\,. \tag{4.7}$$

With similar methods as exposed in the non-discrete case in section 3 we obtain $L^\infty(L^2)$ estimates for $(I_p^k\sigma_\mu, I_p^k\xi_\mu)$

$$\begin{aligned} &\|I_p^k\sigma_\mu\|_{L^\infty(L^2)} \le Const \\ &\|I_p^k\xi_\mu^m\|_{L^\infty(L^2)} \le Const\,. \end{aligned} \tag{4.8}$$

As in section 3 choose $(\sigma_\mu^m - \tau^m, \xi_\mu^m - \eta^m)$ in equation (4.2) and then argue as in [Joh76].

**3)** Let $\overline{\sigma} = \sigma_\mu - \tau$ and $\overline{\xi} = \xi_\mu - \eta$, where $(\tau, \eta)$ satisfies the safe load condition. Test the discretized equation (4.2) with $D_t^{-k}(\overline{\sigma}^m, \overline{\xi}^m)$.

$$\big(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\overline{\sigma}^m\big) + \big(HD_t^{-k}\xi_\mu^m, D_t^{-k}\overline{\xi}^m\big) + \big(G'_\mu((\sigma_\mu^m, \xi_\mu^m)), D_t^{-k}(\overline{\sigma}^m, \overline{\xi}^m)\big) = 0$$

sorting terms

$$\begin{aligned} \big(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\sigma_\mu^m\big) + \big(HD_t^{-k}\xi_\mu^m, D_t^{-k}\xi_\mu^m\big) + \big(G'_\mu((\sigma_\mu^m, \xi_\mu^m)), D_t^{-k}(\overline{\sigma}^m, \overline{\xi}^m)\big) \\ = \big(AD_t^{-k}\sigma_\mu^m, D_t^{-k}\tau^m\big) + \big(HD_t^{-k}\xi_\mu^m, D_t^{-k}\eta^m\big) \end{aligned} \tag{4.9}$$

using on the righthand side the symmetry of $A$, $H$ and Young's inequality with $0<\rho<\alpha_A,\ 0<\gamma<\alpha_H$, set $c_1:=\alpha_A-\rho,\ c_2:=\alpha_H-\gamma$

$$c_1\|D_t^{-k}\sigma_\mu^m\|^2+c_2\|D_t^{-k}\xi_\mu^m\|^2+\big(G'_\mu((\sigma_\mu^m,\xi_\mu^m)),D_t^{-k}(\overline{\sigma}^m,\overline{\xi}^m)\big)\leq\frac{1}{4\rho}\|AD_t^{-k}\tau^m\|^2+\frac{1}{4\gamma}\|HD_t^{-k}\eta^m\|^2\,. \tag{4.10}$$

Because $(\tau,\eta)$ satisfies the safe load condition we have

$$\begin{aligned}c_1\|D_t^{-k}\sigma_\mu^m\|^2+c_2\|D_t^{-k}\xi_\mu^m\|^2+\big(G'_\mu((\sigma_\mu^m,\xi_\mu^m)),D_t^{-k}(\sigma_\mu^m,\xi_\mu^m)\big)&\\ \leq Const(\rho)+\big(G'_\mu((\sigma_\mu^m,\xi_\mu^m)),D_t^{-k}(\tau^m,\eta^m)\big)\,.&\end{aligned} \tag{4.11}$$

We can estimate the tested penalty term as follows

$$\begin{aligned}\big(G'_\mu((\sigma_\mu^m,\xi_\mu^m)),D_t^{-k}(\tau^m,\eta^m)\big)&=\int_\Omega G'_\mu((\sigma_\mu^m,\xi_\mu^m)):(D_t^{-k}(\tau^m,\eta^m))\mathrm{d}x\\ &\leq\int_\Omega |G'_\mu((\sigma_\mu^m,\xi_\mu^m))|\cdot|D_t^{-k}(\tau^m,\eta^m)|\mathrm{d}x\\ &\leq\int_\Omega |G'_\mu((\sigma_\mu^m,\xi_\mu^m))|\cdot|(\dot\tau^m,\dot\eta^m)|\mathrm{d}x\\ &\leq C\int_\Omega |G'_\mu((\sigma_\mu^m,\xi_\mu^m))|\mathrm{d}x\,.\end{aligned} \tag{4.12}$$

Hence

$$c_1\|D_t^{-k}\sigma_\mu^m\|^2+c_2\|D_t^{-k}\xi_\mu^m\|^2+\big(G'_\mu((\sigma_\mu^m,\xi_\mu^m)),D_t^{-k}(\sigma_\mu^m,\xi_\mu^m)\big)\leq C\int_0^t|G'_\mu((\sigma_\mu^m,\xi_\mu^m))|\mathrm{d}x+Const\,. \tag{4.13}$$

Multiply this equation by $k$ and sum over $m=1,\dots,N$

$$\begin{aligned}c_1k\sum_{m=1}^N\|D_t^{-k}\sigma_\mu^m\|^2+c_2k\sum_{m=1}^N\|D_t^{-k}\xi_\mu^m\|^2+\underbrace{\sum_{m=1}^N\big(G'_\mu((\sigma_\mu^m,\xi_\mu^m)),(\sigma_\mu^m-\sigma_\mu^{m-1},\xi_\mu^m-\xi_\mu^{m-1})\big)}_{(*)}&\\ \leq Const\cdot T+Ck\sum_{m=1}^N\int_\Omega|G'_\mu(\sigma_\mu^m,\xi_\mu^m)|\mathrm{d}x\,.&\end{aligned} \tag{4.14}$$

The term $(*)$ is non negative, the inequality for convex differentiable functions yields

$$\int_\Omega G_\mu((\sigma_\mu^m,\xi_\mu^m))-G_\mu((\sigma_\mu^{m-1},\xi_\mu^{m-1}))\mathrm{d}x\leq\big(G'_\mu((\sigma_\mu^m,\xi_\mu^m)),(\sigma_\mu^m-\sigma_\mu^{m-1},\xi_\mu^m-\xi_\mu^{m-1})\big)\,.$$

Summing from $m = 1, \dots, N$ gives a telescope sum

$$(*) = \sum_{m=1}^{N} \big(G'_\mu(\sigma_\mu^m, \xi_\mu^m), (\sigma_\mu^m - \sigma_\mu^{m-1}, \xi_\mu^m - \xi_\mu^{m-1})\big) \geq \sum_{m=1}^{N} \int_\Omega G_\mu((\sigma_\mu^m, \xi_\mu^m)) - G_\mu((\sigma_\mu^{m-1}, \xi_\mu^{m-1}))\mathrm{d}x$$
$$= \int_\Omega G_\mu((\sigma_\mu^N, \xi_\mu^N)) - G_\mu((\sigma_\mu^0, \xi_\mu^0))\mathrm{d}x \geq 0\,. \tag{4.15}$$

By a discrete analogue of theorem 3.2, we have $G'_\mu((\sigma_\mu, \xi_\mu))$ bounded in $L^1(0, T; L^1(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}))$, a choice of $k$ sufficiently small enough yields

$$Ck \sum_{m=1}^{N} \int_\Omega |G'_\mu((\sigma_\mu^m, \xi_\mu^m))|\mathrm{d}x \leq C\|G'_\mu(I_p^k(\sigma_\mu, \xi_\mu))\|_{L^1(L^1)} \leq Const\,.$$

$$k \sum_{m=1}^{N} \|D_t^{-k}\sigma_\mu^m\|^2 + k \sum_{m=1}^{N} \|D_t^{-k}\xi_\mu^m\|^2 \leq Const \cdot T\,.$$

Hence, for $k$ sufficiently small

$$\begin{aligned} &\|I_p^k D_t^{-k}\sigma_\mu\|_{L^2(L^2)} \leq Const \\ &\|I_p^k D_t^{-k}\xi_\mu\|_{L^2(L^2)} \leq Const\,. \end{aligned} \tag{4.16}$$

**4)** The boundedness of $(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m)$ (4.8) and $D_t^{-k}(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m)$ (4.16) yield the existence of $(\tilde{\sigma}_\mu, \tilde{\xi}_\mu)$ and the weak convergene in $L^2(L^2)$

$$\begin{aligned} (I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m) &\rightharpoonup (\tilde{\sigma}_\mu, \tilde{\xi}_\mu) \\ D_t^{-k}(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m) &\rightharpoonup (\dot{\tilde{\sigma}}_\mu, \dot{\tilde{\xi}}_\mu)\,, \end{aligned} \tag{4.17}$$

as $k \to 0$.
We will now show for $\mu$ fixed the strong convergence of $(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m)$: Test the discretized equation (4.2) with $(I_p^k\sigma_\mu^m - \tilde{\sigma}_\mu, I_p^k\xi_\mu^m - \tilde{\xi}_\mu)$

$$0 = \big(AD_t^{-k}I_p^k\sigma_\mu^m, I_p^k\sigma_\mu^m - \tilde{\sigma}_\mu\big) + \big(HD_t^{-k}I_p^k\xi_\mu^m, I_p^k\xi_\mu^m - \tilde{\xi}_\mu\big) + \big(G'_\mu(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m), (I_p^k\sigma_\mu^m - \tilde{\sigma}_\mu, I_p^k\xi_\mu^m - \tilde{\xi}_\mu)\big)\,. \tag{4.18}$$

Note that, on can intercange the interpolation operation with the nonlinearity in $G'_\mu$, since $I_p^k$ is piece wise constant.

We set $z^m := I_p^k\sigma_\mu^m - \tilde{\sigma}_\mu,\ g^m := I_p^k\xi_\mu^m - \tilde{\xi}_\mu$ and using the monotonicity of the penalty term $G'_\mu(I - p^k\sigma_\mu^m, I_p^k\xi_\mu^m)$ we obtain

$$o(1) = \big(AD_t^{-k}z^m, z^m,z^m\big)+\big(HD_t^{-k}g^m,g^m\big)+\underbrace{\big(G'_\mu((I-p^k\sigma_\mu^m, I_p^k\xi_\mu^m)) - G'_\mu((\tilde{\sigma}_\mu, \tilde{\xi}_\mu)),(z^m, g^m)\big)}_{\geq 0}\ . \tag{4.19}$$

Using Young's inequality

$$\begin{aligned} o(1) \geq \frac{1}{2k}\big(Az^m,z^m\big) - \frac{1}{2k}\big(Az^{m-1},z^{m-1}\big) \\ + \frac{1}{2k}\big(Hg^m,g^m\big) - \frac{1}{2k}\big(Hg^{m-1},g^{m-1}\big)\ . \end{aligned} \tag{4.20}$$

Summing over $m = 0, \ldots, l \leq N$ we have

$$o(1) \geq \frac{1}{2k}\big(Az^l,z^l\big) + \frac{1}{2k}\big(Hg^l,g^l\big)\ . \tag{4.21}$$

Where we have use that we define $(I_p^k\sigma^m, \xi^m)(s) = 0$ for $0 \leq s \leq \delta$, in order to satisfy the initial value.
This gives the strong convergene of $(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m) \to (\tilde{\sigma}_\mu, \tilde{\xi}_\mu)$ and as $k \to 0$ by fixed viscosity coefficient $\mu$.
**5)** We have now to show, that $(\tilde{\sigma}_\mu, \tilde{\xi}_\mu)$ are solutions of the penalized hardening problem (2.3). Test the discretized equation (4.2) with $(I_p^k\sigma_\mu^m - \tau, I_p^k\xi_\mu^m - \eta)$, where $(\tau, \eta)$ satisfies the safe load condition.

$$0 = \big(AD_t^{-k}I_p^k\sigma_\mu^m,I_p^k\sigma_\mu^m-\tau\big)+\big(HD_t^{-k}I_p^k\xi_\mu^m,I_p^k\xi_\mu^m-\eta\big)+\big(G'_\mu((I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m)),(I_p^k\sigma_\mu^m-\tau, I_p^k\xi_\mu^m-\eta)\big) \tag{4.22}$$

Due to the strong convergence of $(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m) \to (\tilde{\sigma}_\mu, \tilde{\xi}_\mu)$ and the weak convergence $D_t^{-k}(I_p^k\sigma_\mu^m, I_p^k\xi_\mu^m) \to (\dot{\tilde{\sigma}}_\mu, \dot{\tilde{\xi}}_\mu)$ we get

$$0 = \big(A\dot{\tilde{\sigma}}_\mu,\tilde{\sigma}_\mu - \tau\big) + \big(H\dot{\tilde{\xi}}_\mu,\tilde{\xi}_\mu - \eta\big) + \big(G'_\mu((\tilde{\sigma}_\mu, \tilde{\xi}_\mu)),(\tilde{\sigma}_\mu - \tau, \tilde{\xi}_\mu - \eta)\big)\ . \tag{4.23}$$

□

# 5 Pointwise a.e. equation for the penalized model

The estimates of the last section allow us to show that for fixed viscosity coefficient $\mu$ the strain tensor $\varepsilon(v_\mu)$ is a square integrable function.

**Theorem 5.1** *For fixed viscosity coefficient $\mu$ we have*

$$\begin{aligned} \|\varepsilon(v_\mu)\|_{L^2(L^2)} &\leq Const(\mu) \\ \|v_\mu\|_{L^2(H^1)} &\leq Const(\mu)\,. \end{aligned} \tag{5.1}$$

**proof** Consider the weak formulation (2.3) of penalized hardening.

$$\big(A\dot\sigma_\mu, \chi-\sigma_\mu\big)+\big(H\dot\xi_\mu,\omega-\xi_\mu\big)+\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\chi-\sigma_\mu,\omega-\xi_\mu)\big)+\langle v_\mu,\operatorname{div}(\chi-\sigma_\mu)\rangle=0$$

Choose $(\chi,\omega)\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))\times L^2(0,T;L^2(\Omega,\mathbb{R}^m))$ with

$$\left.\begin{aligned} -\operatorname{div}\chi &= f \text{ in } \Omega \\ \chi\cdot\vec n &= p \text{ on } \Gamma_N \\ \omega &= \xi_\mu\,. \end{aligned}\right\} \tag{5.2}$$

Inserting $(\chi,\omega)$ into the weak formulation results

$$\big(A\dot\sigma_\mu,\chi-\sigma_\mu\big)+\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\chi-\sigma_\mu,0)\big)=0\,. \tag{5.3}$$

The space of divergence free tensor fields is the orthogonal complement of the space of strain tensors ([Tem85] see e.g. also [Löb07a]). This implies

$$A\dot\sigma_\mu+\frac{\partial}{\partial\sigma_\mu}G_\mu((\sigma_\mu,\xi_\mu))=\varepsilon(v_\mu) \tag{5.4}$$

pointwise almost everywhere in $\Omega$ for $t$ fixed. We have $\varepsilon(v_\mu)\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ and

$$\big(A\dot\sigma_\mu,\chi-\sigma_\mu\big)+\big(H\dot\xi_\mu,\omega-\xi_\mu\big)+\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\chi-\sigma_\mu,\omega-\xi_\mu)\big)-\big(\varepsilon(u_\mu),\chi-\sigma_\mu\big)=0\,. \tag{5.5}$$

The linear growth condition (2.2) and $\|\sigma_\mu\|_{L^\infty(L^2)}$, $\|\xi_\mu\|_{L^\infty(L^2)}\leq Const$ gives the estimate

$$\|G'_\mu((\sigma_\mu,\xi_\mu))\|_{L^2(L^2)}\leq\frac{1}{\mu}Const\,. \tag{5.6}$$

Together with Korn's inequality

$$\begin{aligned} \|\varepsilon(v_\mu)\|_{L^2(L^2)} &\leq \frac{1}{\mu}Const \\ \|v_\mu\|_{L^2(H^1)} &\leq \frac{1}{\mu}Const\,. \end{aligned}$$

□

This leads to the pointwise almost everywhere equation

$$\begin{pmatrix} A\dot{\sigma}_\mu \\ H\dot{\xi}_\mu \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial\sigma_\mu} G_\mu(\sigma_\mu, \xi_\mu) \\ \frac{\partial}{\partial\xi_\mu} G_\mu(\sigma_\mu, \xi_\mu) \end{pmatrix} = \begin{pmatrix} \varepsilon(v_\mu) \\ 0 \end{pmatrix} . \tag{5.7}$$

We obtain a formulation containing the strain tensor $\varepsilon(v_\mu)$.
Find $\big((\sigma_\mu, \xi_\mu), v_\mu\big)$ such that for all $(\tau, \eta) \in \mathcal{M}$ holds

$$\big(A\dot{\sigma}_\mu, \tau - \sigma_\mu\big) + \big(H\dot{\xi}_\mu, \eta - \xi_\mu\big) + \big(G'_\mu((\sigma_\mu, \xi_\mu)), (\tau - \sigma_\mu, \eta - \xi_\mu)\big) - \big(\varepsilon(v_\mu), \tau - \sigma_\mu\big) = 0 \tag{5.8}$$

# 6 Convergence of the penalized problem to the hardening model

We will now proof the convergence of the stress tensor and hardening parameter $(\sigma_\mu, \xi_\mu)$ to the solution of the initial hardening problem (1.2).

**Theorem 6.1** *There exists a subsequence of the sequence* $(\sigma_\mu, \xi_\mu)$ *of solutions of the penalized hardening problem* (2.3)*, converging weakly in* $L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym}))\times L^2(0,T;L^2(\Omega;\mathbb{R}^m))$ *to the solution of the initial hardening problem* (1.2)*.*

**proof** From theorems 3.1 and 4.1 we know that the sequence of the stress tensor, hardening parameters and their time derivatives is uniformly bounded in $L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))\times L^2(0,T;L^2(\Omega;\mathbb{R}^m))$.
Extracting a suitable subsequence $(\sigma_{\mu_l}, \xi_{\mu_l})$, there exists $(\tilde\sigma, \tilde\xi) \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))\times L^2(0,T;L^2(\Omega;\mathbb{R}^m))$ such that for $\mu_l \to 0$

$$\begin{aligned} &(\sigma_{\mu_l}, \xi_{\mu_l}) \rightharpoonup (\tilde\sigma, \tilde\xi) \\ &(\dot\sigma_{\mu_l}, \dot\xi_{\mu_l}) \rightharpoonup (\dot{\tilde\sigma}, \dot{\tilde\xi})\,. \end{aligned} \tag{6.1}$$

Define

$$\mathcal{X} := \mathcal{M}\cap\big\{(\tau,\eta)\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))\times L^2(0,T;L^2(\Omega;\mathbb{R}^m))\,|\,-\operatorname{div}\tau = f \text{ in } \Omega\times[0,T]\big\}$$

Test the pointwise almost everywhere penalized hardening law (5.8) with $(\tau - \sigma_{\mu_l}, \eta - \xi_{\mu_l})$ where $(\tau,\eta)\in\mathcal{K}\cap\mathcal{X}$. Then we have a.e. on $[0,T]$

$$\big(A\dot\sigma_{\mu_l}, \tau - \sigma_{\mu_l}\big) + \big(H\dot\xi_{\mu_l}, \eta - \xi_{\mu_l}\big) + \big(G'_\mu((\sigma_{\mu_l}, \xi_{\mu_l})), (\tau - \sigma_{\mu_l}, \eta - \xi_{\mu_l})\big) = 0\,. \tag{6.2}$$

The tested penalty term is non positive

$$\big(G'_\mu((\sigma_{\mu_l}, \xi_{\mu_l})), (\tau - \sigma_{\mu_l}, \eta - \xi_{\mu_l})\big) \le \underbrace{G_\mu(\tau,\eta)}_{=0} - \underbrace{G_\mu(\sigma_{\mu_l}, \xi_{\mu_l})}_{\ge 0}\,.$$

This yields

$$\big(A\dot\sigma_{\mu_l}, \tau - \sigma_{\mu_l}\big) + \big(H\dot\xi_{\mu_l}, \eta - \xi_{\mu_l}\big) \ge 0\,. \tag{6.3}$$

From (6.3) we infer (note $(\sigma_{\mu_l}, \xi_{\mu_l})(0) = 0$)

$$\begin{aligned} 0 &\le \limsup_{\mu_l\to 0}\left[-\int_0^t \big(A\dot\sigma_{\mu_l}, \sigma_{\mu_l}\big)\mathrm{d}s + \int_0^t \big(A\dot\sigma_{\mu_l}, \tau\big)\mathrm{d}s - \int_0^t \big(H\dot\xi_{\mu_l}, \xi_{\mu_l}\big)\mathrm{d}s + \int_0^t \big(H\dot\xi_{\mu_l}, \eta\big)\mathrm{d}s\right] \\ &\le \limsup_{\mu_l\to 0}\left[-\frac{1}{2}\big(A\sigma_{\mu_l}, \sigma_{\mu_l}\big)(t) - \frac{1}{2}\big(H\xi_{\mu_l}, \xi_{\mu_l}\big)(t) + \int_0^t \big(A\dot\sigma_{\mu_l}, \tau\big)\mathrm{d}s + \int_0^t \big(H\dot\xi_{\mu_l}, \eta\big)\mathrm{d}s\right]\,. \end{aligned} \tag{6.4}$$

We average with respect to $t$ on an intervall $(s,s+h)$ and use that

$$-\frac{1}{2h}\int_s^{s+h}\big(A\sigma_{\mu_l},\sigma_{\mu_l}\big)(t)\mathrm{d}t-\frac{1}{2h}\int_s^{s+h}\big(H\xi_{\mu_l},\xi_{\mu_l}\big)(t)\mathrm{d}t$$

are upper semicontinuous in the weak topology on $L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))$, thus

$$\begin{aligned}(6.4)\leq\limsup_{\mu_l\to 0}\Big[&-\frac{1}{2h}\int_s^{s+h}\big(A\sigma_{\mu_l},\sigma_{\mu_l}\big)(t)\mathrm{d}t-\frac{1}{2h}\int_s^{s+h}\big(H\xi_{\mu_l},\xi_{\mu_l}\big)(t)\mathrm{d}t\\&+\frac{1}{h}\int_s^{s+h}\int_0^t\big(A\dot{\sigma}_{\mu_l},\tau\big)\mathrm{d}s\mathrm{d}t+\frac{1}{h}\int_s^{s+h}\int_0^t\big(H\dot{\xi}_{\mu_l},\eta\big)\mathrm{d}s\mathrm{d}t\Big]\end{aligned}\tag{6.5}$$

By the the weak convergence (6.1) and $(\sigma_\mu,\xi_\mu)(0)=0$ we deduce almost everywhere in $[0,T]$

$$0\leq\left[-\frac{1}{2h}\int_s^{s+h}\big(A\tilde{\sigma},\tilde{\sigma}\big)\mathrm{d}t-\frac{1}{2h}\int_s^{s+h}\big(H\tilde{\xi},\tilde{\xi}\big)\mathrm{d}t+\frac{1}{h}\int_s^{s+h}\int_0^t\big(A\dot{\tilde{\sigma}},\tau\big)\mathrm{d}s\mathrm{d}t+\frac{1}{h}\int_s^{s+h}\int_0^t\big(H\dot{\tilde{\xi}},\eta\big)\mathrm{d}s\mathrm{d}t\right].\tag{6.6}$$

The weak limit $\tilde{\sigma}$ satisfies $\tilde{\sigma}(0)=0$, since $\sigma_\mu,\dot{\sigma}_\mu$ are uniform bounded in $L^\infty(L^2),L^2(L^2)$ with respect to the viscosity coefficient $\mu$.
Now, we let $h\to 0$, and a theorem about Lebesgue points provides for almost all $t\in[0,T)$

$$0\leq\int_0^t\big(A\dot{\tilde{\sigma}},\tau-\tilde{\sigma}\big)\mathrm{d}s+\int_0^t\big(H\dot{\tilde{\xi}},\eta-\tilde{\xi}\big)\mathrm{d}s\,.\tag{6.7}$$

Since $(\sigma_{\mu_l},\xi_{\mu_l})\in\mathcal{X}$ and $\mathcal{X}$ is closed and convex the weak limit satisfies $(\tilde{\sigma},\tilde{\xi})\in\mathcal{X}$. We have to show that $(\tilde{\sigma},\tilde{\xi})\in\mathcal{K}$. By theorem 3.2 we have

$$\int_0^T\big(F(\sigma_{\mu_l},\xi_{\mu_l})\big)_+^2\mathrm{d}s\leq\mu_l\cdot Const\,.$$

By letting $\mu_l\to 0$, we obtain $(\tilde{\sigma},\tilde{\xi})\in\mathcal{K}$ and this proofs the statement of the theorem. By standard techniques we obtain the convergence of the whole sequence $(\sigma_\mu,\xi_\mu)$. □

Now we show the convergence of the displacement velocity $v_\mu$.

**Theorem 6.2** *There exists a subsequence $(v_{\mu_l})$ converging weakly in $L^2(0,T;H^1_{\Gamma_D}(\Omega,\mathbb{R}^n))$ to $v$ displacement velocity solution of the initial hardening problem* (1.2)*.*

**proof** We will make use of results from the next section. By theorems 7.1, 7.3 in section 7 we know that the sequence $(v_\mu)$ is uniformly bounded in $L^2(0,T;H^1_{\Gamma_D}(\Omega,\mathbb{R}^n))$. Thus there exists a $\hat{v} \in L^2(0,T;H^1_{\Gamma_D}(\Omega,\mathbb{R}^n))$ and a suitable subsequence $\big((\sigma_{\mu_l},\xi_{\mu_l}),v_{\mu_l}\big)$ such that for $\mu_l \to 0$

$$\big((\sigma_{\mu_l},\xi_{\mu_l}),v_{\mu_l}\big) \rightharpoonup \big((\sigma,\xi),\hat{v}\big)\,.$$

We have to show that $\hat{v}$ is a solution of (1.2).
Test equation (5.8) with $(\sigma_\mu - \tau, \xi_\mu - \eta)$ where $(\tau,\eta) \in \mathcal{M}$. We have like in theorem 6.1 by using the balance of forces (2.4)

$$-\big(v_{\mu_l}, \operatorname{div}(\sigma_{\mu_l} - \tau)\big) \geq \big(A\dot{\sigma}_{\mu_l}, \sigma_{\mu_l} - \tau\big) + \big(H\dot{\xi}_{\mu_l}, \xi_{\mu_l} - \eta\big) \text{ a.e. in } [0,T]\,. \tag{6.8}$$

Arguing as in the preceeding theorem we see that $\hat{v}$ is the displacement velocity solution to the initial hardening problem (1.2). □

**Theorem 6.3** *The sequence $(\sigma_\mu,\xi_\mu)$ converges strongly in $L^2(L^2)$.*

**proof** Test (5.8) with $(\sigma - \sigma_\mu, \xi - \xi_\mu)$, where $(\sigma,\xi)$ is a solution of (1.2)

$$\big(A\dot{\sigma}_\mu, \sigma - \sigma_\mu\big) + \big(H\dot{\xi}_\mu, \xi - \xi_\mu\big) + \big(G'_\mu(\sigma_\mu,\xi_\mu), (\sigma - \sigma_\mu, \xi - \xi_\mu)\big) = 0\,. \tag{6.9}$$

Due to the monotonicity of $G'_\mu$ the tested penalty term is non negative. Hence

$$\big(A(\dot{\sigma} - \dot{\sigma}_\mu), \sigma - \sigma_\mu\big) + \big(H(\dot{\xi} - \dot{\xi}_\mu), \xi - \xi_\mu\big) \leq \big(A\dot{\sigma}, \sigma - \sigma_\mu\big) + \big(H\dot{\xi}, \xi - \xi_\mu\big)\,. \tag{6.10}$$

Integration of (6.10) from 0 to $t$ yields

$$\frac{1}{2}\big(A(\sigma - \sigma_\mu), \sigma - \sigma_\mu\big) + \frac{1}{2}\big(H(\xi - \xi_\mu), \xi - \xi_\mu\big) \leq \int_0^t \big(A\dot{\sigma}, \sigma - \sigma_\mu\big)\mathrm{d}s + \int_0^t \big(H\dot{\xi}, \xi - \xi_\mu\big)\mathrm{d}s\,. \tag{6.11}$$

Passing to the limit $\mu \to 0$ gives

$$\liminf_{\mu\to 0} \frac{1}{2}\Big[\big(A(\sigma - \sigma_\mu), \sigma - \sigma_\mu\big) + \big(H(\xi - \xi_\mu), \xi - \xi_\mu\big)\Big] \leq \underbrace{\lim_{\mu\to 0}\left[\int_0^t \big(A\dot{\sigma}, \sigma - \sigma_\mu\big)\mathrm{d}s + \int_0^t \big(H\dot{\xi}, \xi - \xi_\mu\big)\mathrm{d}s\right]}_{=0} \tag{6.12}$$

since $(\sigma_\mu,\xi_\mu) \rightharpoonup (\sigma,\xi)$. □

# 7 $H^1$-Regularity

In this section we show the regularity of the hardening model. Johnson ([Joh78]) was the first to show the $H^1$-regularity for the displacement velocities. He used as approximation the projection onto the set of all stress and hardening parameter satisfying the yield criterion.(see e.g. Duvaut Lions [DL76] for projection models.) Seregin [Ser94] even obtained $H^2_{loc}$ regularity results for the displacements (not velocities!) in kinematic hardening.
We show the uniform boundedness for $\varepsilon(v_\mu)$ and then we regard the stress tensor. The proof in the present paper looks simpler, although it is, of course, similar. The regularity problem for the displacement velocities was also discussed in [Pai02].
We assume for $f$:

$$\left.\begin{aligned} f &\in L^\infty(0,T;L^n(\Omega,\mathbb{R}^n)) \\ Df &\in L^\infty(0,T;L^n_{loc}(\Omega,\mathbb{R}^{n\times n})) \\ \triangle f &\in L^\infty(0,T;L^n_{loc}(\Omega,\mathbb{R}^n)) \end{aligned}\right\} \tag{7.1}$$

## 7.1 isotropic hardening

**Theorem 7.1** *Let* (7.1) *and the safe load confdition* (3.1) *hold true, then the strain tensor* $\varepsilon(v_\mu)$ *of penalized hardening is uniformly bounded in* $L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym}))$.

**proof** Let $\omega \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ with $\operatorname{div}\omega \in L^\infty(0,T;L^n(\Omega,\mathbb{R}^n))$ and $\|\omega\|_{L^2(L^2)} = 1$. Consider the test function

$$\begin{aligned} \tau &= \omega \\ \eta &= |\omega_D| - \kappa\,. \end{aligned} \tag{7.2}$$

test the weak formulation (5.8) with $(\sigma_\mu - \tau, \xi_\mu - \eta)$ where $(\sigma_\mu, \xi_\mu)$ are solutions of the penalized equation.
We examine the tested penalty term $\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big)$

$$\begin{aligned} &\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big) \\ &= \frac{1}{\mu}\int_\Omega \big(|\sigma_{\mu D}|-(\kappa+\xi_\mu)\big)_+ \left(\frac{\sigma_{\mu D}}{|\sigma_{\mu D}|} : (\sigma_\mu - \omega)\right)\mathrm{d}x - \frac{1}{\mu}\int_\Omega \big(|\sigma_{\mu D}|-(\kappa+\xi_\mu)\big)_+ (\xi_\mu - |\omega_D| + \kappa)\mathrm{d}x \\ &\overset{\text{Cauchy Schwarz}}{\geq} \frac{1}{\mu}\int_\Omega \big(|\sigma_{\mu D}|-(\kappa+\xi_\mu)\big)_+ (|\sigma_{\mu D}|-|\omega_D|)\mathrm{d}x - \frac{1}{\mu}\int_\Omega \big(|\sigma_{\mu D}|-(\kappa+\xi_\mu)\big)_+ (\xi_\mu - |\omega_D| + \kappa)\mathrm{d}x \\ &= \frac{1}{\mu}\int_\Omega \big(|\sigma_{\mu D}| - (\kappa+\xi_\mu)\big)_+ (|\sigma_{\mu D}| - (\kappa+\xi_\mu)) \geq 0\,. \end{aligned} \tag{7.3}$$

The definiteness of the tested penalty term yields the inequality

$$\big(A\dot{\sigma}_\mu,\sigma_\mu-\omega\big)+\big(H\dot{\xi}_\mu,\xi_\mu-(|\omega_D|-\kappa)\big)\le\big(\varepsilon(v_\mu),\sigma_\mu-\omega\big) \tag{7.4}$$

sorting terms

$$\big(\varepsilon(v_\mu),\omega\big)\le-\big(A\dot{\sigma}_\mu,\sigma_\mu\big)+\big(A\dot{\sigma}_\mu,\omega\big)-\big(H\dot{\xi}_\mu,\xi_\mu\big)+\big(H\dot{\xi}_\mu,|\omega_D|\big)-\big(H\dot{\xi}_\mu,\kappa\big)+\big(\varepsilon(v_\mu),\sigma_\mu\big)\,. \tag{7.5}$$

Using the balance of forces (2.4) $\big(\varepsilon(v_\mu),\sigma_\mu\big)=-\big(v_\mu,f\big)+\int_{\Gamma_N}pv_\mu\mathrm{d}\Gamma$ and integrate (7.5) from 0 to $t$

$$\begin{aligned}\int_0^t\big(\varepsilon(v_\mu),\omega\big)\mathrm{d}s\le&-\frac{1}{2}\big(A\sigma_\mu,\sigma_\mu\big)-\frac{1}{2}\big(H\xi_\mu,\xi_\mu\big)+\int_0^t\big(A\dot{\sigma}_\mu,\omega\big)\mathrm{d}s\\&+\int_0^t\big(H\dot{\xi}_\mu,|\omega_D|\big)\mathrm{d}s-\int_0^t\big(H\dot{\xi}_\mu,\kappa\big)\mathrm{d}s-\int_0^t\big(v_\mu,f\big)\mathrm{d}s+\int_0^t\int_{\Gamma_N}pv_\mu\mathrm{d}\Gamma\mathrm{d}s\,.\end{aligned} \tag{7.6}$$

The Hölder and Young inequality gives with use of $|\omega_D|\le|\omega|$ almost everywhere the estimate

$$\begin{aligned}\int_0^t\big(\varepsilon(v_\mu),\omega\big)\mathrm{d}s\le&\frac{1}{2}\big(A\sigma_\mu,\sigma_\mu\big)+\frac{1}{2}\big(H\xi_\mu,\xi_\mu\big)+\int_0^t\|A\dot{\sigma}_\mu\|^2\mathrm{d}s+2\int_0^t\|\omega\|^2\mathrm{d}s+2\int_0^t\|H\dot{\xi}_\mu\|^2\mathrm{d}s\\&+\int_0^t\|v_\mu\|_{L^{\frac{n}{n-1}}}\cdot\|f\|_{L^n}\mathrm{d}s+\int_0^t\|v_\mu\|_{L^1(\Gamma_N)}\cdot\|p\|_{L^\infty(\Gamma_N)}\mathrm{d}s+t\cdot\kappa^2|\Omega|\le Const\,.\end{aligned} \tag{7.7}$$

We have $\|\varepsilon(v_\mu)\|_{L^2(L^2)}\le Const$. By Korn's inequality we obtain $\|v_\mu\|_{L^2(H^1)}\le Const\,.$ □

**Theorem 7.2** *With the assumptions of theorem 7.1, the strain tensor $\varepsilon(v)$ with $v$ solution of isotropic hardening holds*

$$\begin{aligned}\varepsilon(v)&\in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym}))\\v&\in L^2(0,T;H^1_{\Gamma_D}(\Omega,\mathbb{R}^n))\,.\end{aligned} \tag{7.8}$$

**proof** The convergence of $\big((\sigma_\mu,\xi_\mu),v_\mu\big)$ to $\big((\sigma,\xi),v\big)$ yields the statement. □

## 7.2 kinematic hardening

**Theorem 7.3** *Under the assumptions of theorem 7.1, the strain tensor* $\varepsilon(v_\mu)$ *of penalized kinematic hardening holds*

$$\|\varepsilon(v_\mu)\|_{L^2(L^2)} \leq Const\,.$$

**proof** Let $\omega \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ with $\operatorname{div}\omega \in L^\infty(0,T;L^n(\Omega,\mathbb{R}^n))$ and $\|\omega\|_{L^2(L^2)} = 1$. Consider the testfunction

$$\begin{aligned} \tau &= \omega \\ \eta &= \xi_\mu - \sigma_\mu + \omega\,. \end{aligned} \tag{7.9}$$

Test equation (5.8) with $(\sigma_\mu - \tau, \xi_\mu - \eta)$ where $(\sigma_\mu, \xi_\mu)$ are solutions.
The tested penalty term $\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big)$ takes the following form

$$\begin{aligned} &\big(G'_\mu((\sigma_\mu,\xi_\mu)),(\sigma_\mu-\tau,\xi_\mu-\eta)\big) \\ &= \frac{1}{\mu}\int_\Omega \frac{\big(|\sigma_{\mu D}-\xi_{\mu D}|-\kappa\big)_+}{|\sigma_{\mu D}-\xi_{\mu D}|}\Big((\sigma_{\mu D}-\xi_{\mu D}):(\sigma_\mu-\omega)+(\xi_{\mu D}-\sigma_{\mu D}):(\sigma_\mu-\omega)\Big) \\ &= 0\,. \end{aligned} \tag{7.10}$$

Thus we have the inequality

$$\big(A\dot\sigma_\mu,\sigma_\mu-\omega\big) + \big(H\dot\xi_\mu,\sigma_\mu-\omega\big) \leq \big(\varepsilon(v_\mu),\sigma_\mu-\omega\big) \tag{7.11}$$

sorting terms

$$\big(\varepsilon(v_\mu),\omega\big) \leq \big(A\dot\sigma_\mu,\sigma_\mu\big) + \big(A\dot\sigma_\mu,\omega\big) - \big(H\dot\xi_\mu,\sigma_\mu\big) + \big(H\dot\xi_\mu,\omega\big) + \big(\varepsilon(v_\mu),\sigma_\mu\big)\,. \tag{7.12}$$

Integrate from 0 to $t$ and using the Young- and Hölder inequality.
With $\big(\varepsilon(v_\mu),\sigma_\mu\big) = -\big(v_\mu,f\big) + \int_{\Gamma_N} p v_\mu \mathrm{d}\Gamma$ we have

$$\begin{aligned} \int_0^t \big(\varepsilon(v_\mu),\omega\big)\mathrm{d}s &\leq \frac{1}{2}\big(A\sigma_\mu,\sigma_\mu\big) + \int_0^t \|A\dot\sigma_\mu\|^2\mathrm{d}s + \int_0^t \|\omega\|^2\mathrm{d}s + 2\int_0^t \|H\dot\xi_\mu\|^2\mathrm{d}s \\ &\quad + 2\int_0^t \|\sigma_\mu\|^2\mathrm{d}s + \int_0^t \|f\|_{L^n}\cdot\|v_\mu\|_{L^{\frac{n}{n-1}}}\mathrm{d}s + \int_0^t \|v_\mu\|_{L^1(\Gamma_N)}\cdot\|p\|_{L^\infty(\Gamma_N)}\mathrm{d}s \leq Const\,. \end{aligned} \tag{7.13}$$

Thus $\varepsilon(v_\mu)$ is bounded in $L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\text{sym}}))$ independent of the viscosity coefficient $\mu$.
□

**Theorem 7.4** *We have for the strain tensor $\varepsilon(v)$, with $v$ solution of kinematic hardening $\varepsilon(v) \in L^2(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{sym})$ and $v \in L^2(0,T;H^1_{\Gamma_D}(\Omega,\mathbb{R}^n))$.*

**proof** The convergence of $\big((\sigma_\mu,\xi_\mu),v_\mu\big)$ to $\big((\sigma,\xi),v\big)$ yields the statement by using theorem 7.3 . □

## 7.3 Local differentiability of the stress tensor and hardening parameter

With the uniform boundedness of the strain tensor $\varepsilon(v_\mu)$ we are now able to show the local differentiability of the stress tensor $\sigma$ and hardening parameter $\xi$. Seregin [Ser94] obtained the local differentiability for the stresses and hardening parameters with a dual method, without using a penalty approximation of the variational inequality (1.2).

**Remark:** To prove the differentiability we need more regularity for the elasticity tensor $A$ and hardening modulus $H$: For the sake of brevity we take $A \in \hom(\mathbb{R}^{n\times n}_{\text{sym}},\mathbb{R}^{n\times n}_{\text{sym}})$ constant[1]. The same assumption holds for the hardening modulus $H$.

**Theorem 7.5** *With the assumptions of theorem 7.1 and the remark above we have: In the case of isotropic-,kinematic hardening the stress tensor and hardening parameter holds $\sigma \in L^2(0,T;H^1_{loc}(\Omega,\mathbb{R}^{n\times n}_{sym})), \xi \in L^2(0,T;H^1_{loc}(\Omega,\mathbb{R}^m))$ with $m=1$ or $m=n\times n$.*

**proof** Let $\theta \in C^\infty_o(\Omega)$ be a cutoff function and $0 < h < \frac{1}{2}\operatorname{dist}(\operatorname{supp}\theta,\partial\Omega)$. Test the penalized equation (5.8) with $-D_j^{-h}(\theta^2 D_j^h(\sigma_\mu,\xi_\mu))$.

$$\big(\theta A D_j^h\dot\sigma_\mu,\theta D_j^h\sigma_\mu\big)+\big(H\theta D_j^h\dot\xi_\mu,\theta D_j^h\xi_\mu\big)+\big(\theta D_j^h G'_\mu((\sigma_\mu,\xi_\mu)),\theta D_j^h(\sigma_\mu,\xi_\mu)\big) = \big(D_j^h\varepsilon(v_\mu),\theta^2 D_j^h\sigma_\mu\big) \tag{7.14}$$

By monotonicity, the tested penaltyterm reads

$$\big(\theta D_j^h G'_\mu((\sigma_\mu,\xi_\mu)),\theta D_j^h(\sigma_\mu,\xi_\mu)\big) \geq 0\,.$$

[1]For example one can take the inverse Lamé-Navier Operator $A\sigma = \frac{1}{2\mu_0}\sigma - \frac{\lambda_0}{3\lambda_0+2\mu_0}\operatorname{tr}(\sigma)\cdot Id$, where $\lambda_0,\mu_0 > 0$ are the Lamé constants of the material.

write

$$\big(\theta A D^h_j \dot\sigma_\mu, \theta D^h_j \sigma_\mu\big) = \frac{1}{2}\frac{d}{d\,t}\big(\theta A D^h_j \sigma_\mu, \theta D^h_j \sigma_\mu\big)$$
$$\big(H\theta D^h_j \dot\xi_\mu, \theta D^h_j \xi_\mu\big) = \frac{1}{2}\frac{d}{d\,t}\big(H\theta D^h_j \xi_\mu, \theta D^h_j \xi_\mu\big)\,.$$

Integrate equation (7.14) from 0 to $t$ and using the ellipticity of $A$, $H$ we obtain

$$\frac{\alpha_A}{2}\|\theta D^h_j \sigma_\mu\|^2 + \frac{\alpha_H}{2}\|\theta D^h_j \xi_\mu\|^2 \le \int_0^t \big(D^h_j \varepsilon(v_\mu), \theta^2 D^h_j \sigma_\mu\big)\mathrm{d}s\,. \tag{7.15}$$

Using the balance of forces (2.4) on the left hand side yields

$$\int_0^t \big(D^h_j \varepsilon(v_\mu), \theta^2 D^h_j \sigma_\mu\big)\mathrm{d}s = -\int_0^t \big(D^h_j v_\mu, \operatorname{grad}\theta^2 D^h_j \sigma_\mu\big)\mathrm{d}s - \int_0^t \big(D^h_j v_\mu, \theta^2 D^h_j f\big)\mathrm{d}s\,.$$

We know that $\varepsilon(v_\mu)$ and $v$ are uniform bounded, thus we obtain for $-\int_0^t \big(D^h_j v_\mu, \operatorname{grad}\theta^2 D^h_j \sigma_\mu\big)\mathrm{d}s$ and $h$ small enough

$$-\int_0^t \big(D^h_j v_\mu, \operatorname{grad}\theta^2 D^h_j \sigma_\mu\big)\mathrm{d}s \le \gamma C_\theta \int_0^t \|\theta D^h_j \sigma_\mu\|^2 \mathrm{d}s + \frac{1}{4\gamma}\int_0^t \|v_\mu\|^2_{H^1}\mathrm{d}s$$
$$\le \gamma C_\theta \int_0^t \|\theta D^h_j \sigma_\mu\|^2 \mathrm{d}s + Const\,.$$

We use discrete partial integration for $-\int_0^t \big(D^h_j v_\mu, \theta^2 D^h_j f\big)\mathrm{d}s$ and obtain

$$-\int_0^t \big(D^h_j v_\mu, \theta^2 D^h_j f\big)\mathrm{d}s = \int_0^t \big(v_\mu, D^{-h}_j \theta^2 D^h_j f\big)\mathrm{d}s + \int_0^t \big(v_\mu, E^{-h}_j \theta^2 \triangle^h f\big)\mathrm{d}s \le Const\,.$$

Where $\triangle^h f = D^{-h}_j (D^h_j f)$ denotes the finite difference approximation to the Laplace operator $\triangle f$.

We obtain the inequality

$$\frac{\alpha_A}{2}\|\theta D^h_j \sigma_\mu\|^2 + \frac{\alpha_H}{2}\|\theta D^h_j \xi_\mu\|^2 \le Const + \frac{1}{4\gamma}Const + \gamma C_\theta \int_0^t \|\theta D^h_j \sigma_\mu\|^2 \mathrm{d}s\,.$$

and Gronwall's lemma yields

$$\|\theta D^h_j \sigma_\mu\|^2 \le Const$$
$$\|\theta D^h_j \xi_\mu\|^2 \le Const$$

independent of the viscosity coefficient $\mu$. By passing to the limit one gets the local differentiability for the stress tensor and hardening parameters of the initial hardening problem (1.2). □

# 8 $L^\infty(L^2)$-estimates for $\varepsilon(\dot u_\mu)$ and $(\dot\sigma_\mu,\dot\xi_\mu)$

In this section we derive $L^\infty(L^2)$ estimates for the velocities of the stress tensor and hardening parameters. We also show the existence of the initial value $(\dot\sigma_\mu,\dot\xi_\mu)(0)$.
We do this by giving an estimate of the form

$$\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\big|_T \le \frac{1}{h}\int_0^h\!\!\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\mathrm{d}t + K\,.$$

For the preceding calculations we need to extend the safe load condition (3.1).

**extended safe load:**

There exists a $(\tau,\eta)\in W^{2,\infty}(0,T;L^\infty(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))\times W^{1,\infty}(0,T;L^\infty(\Omega,\mathbb{R}^m))$ and $\delta>0$ such that

$$\left.\begin{aligned} -\operatorname{div}\tau &= f \text{ in } \Omega\times[0,T]\\ \tau\cdot\vec n &= p \text{ on } \Gamma_N\times[0,T]\\ (\tau,\eta)(0) &= 0 \text{ in } \Omega\times\mathbb{R}^m\times\{t=0\}\\ F(\tau,\eta) &\le -\delta < 0\,. \end{aligned}\right\} \tag{8.1}$$

The novell feature in this hypothesis is $\underline{\ddot\tau\in L^\infty(0,T;L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}))}$.

**Theorem 8.1** *Let the assumptions of section 1 and the extended safe load condition* (8.1) *hold true, let $h\in(0,T)$, then there holds the estimate*

$$\int_0^h\!\!\int_\Omega |\varepsilon(\dot u_\mu)|^2\mathrm{d}x\mathrm{d}t \le K\int_0^h\!\!\int_\Omega |\dot\sigma_\mu|^2 + |\dot\xi_\mu|^2\mathrm{d}x\mathrm{d}t. \tag{8.2}$$

**proof** We test the pointwise hardening law (5.7) with $(\varepsilon(\dot u_\mu),|\varepsilon(\dot u_\mu)|)^\intercal$. In both cases, isotropic and kinematic, the tested penalty terms cancel out after the use of the Cauchy Schwarz inequality. Application of Young's inequality gives the desired result. □

**Theorem 8.2** *We have*

$$\frac{1}{h}\int_0^h\!\!\int_\Omega \varepsilon(\dot u_\mu):\dot\sigma_\mu\mathrm{d}x\mathrm{d}t \le \frac{K}{h}\int_0^h\!\!\int_\Omega |\dot\sigma_\mu|^2 + |\dot\xi_\mu|^2\mathrm{d}x\mathrm{d}t + K\,. \tag{8.3}$$

**proof** Let $(\tau,\eta)$ satisfy the safe load condition (8.1). Then

$$\begin{aligned} \frac{1}{h}\int_0^h\!\!\int_\Omega \varepsilon(\dot u_\mu):\dot\sigma_\mu &= \frac{1}{h}\int_0^h\underbrace{\int_\Omega \varepsilon(\dot u_\mu):(\dot\sigma_\mu-\dot\tau)\mathrm{d}x\mathrm{d}t}_{=0} + \frac{1}{h}\int_0^h\!\!\int_\Omega \varepsilon(\dot u_\mu):\dot\tau\mathrm{d}x\mathrm{d}t\\ &\overset{\text{Young}}{\le} K_{\dot\tau} + \frac{\delta}{h}\int_0^h\!\!\int_\Omega |\varepsilon(\dot u_\mu)|^2\mathrm{d}x\mathrm{d}t\\ &\overset{(8.2)}{\le} K_{\dot\tau} + \frac{K}{h}\int_0^h\!\!\int_\Omega |\dot\sigma_\mu|^2 + |\dot\xi_\mu|^2\mathrm{d}x\mathrm{d}t\,. \end{aligned} \tag{8.4}$$

□

**Theorem 8.3** *Under the assumptions of section 1 and the extended safe load condition* (8.1) *there holds the estimate*

$$\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\big|_T \le \frac{1}{h}\int_0^h\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\mathrm{d}t + K\,. \tag{8.5}$$

**proof** Let $0 < \rho < T$ and $t_0 \in (0,\rho)$ and apply the finite difference Operator $D_t^{-k}$, $0 < k \ll t_0$, in time to the pointwise a.e. equation (5.7) and then test with $D_t^{-k}(\sigma_\mu, \xi_\mu)$

$$\int_\Omega D_t^{-k}\varepsilon(\dot u_\mu) : D_t^{-k}\sigma_\mu \mathrm{d}x = \int_\Omega AD_t^{-k}\dot\sigma_\mu : D_t^{-k}\sigma_\mu + HD_t^{-k}\dot\xi_\mu : D_t^{-k}\xi_\mu \mathrm{d}x + \int_\Omega D_t^{-k}G'_\mu(\sigma_\mu,\xi_\mu) : D_t^{-k}(\sigma_\mu,\xi_\mu)\mathrm{d}x\,. \tag{8.6}$$

A zero addition with $D_t^{-k}\tau$, where $\tau$ satisfies the extended safe load condition (8.1), in the $\varepsilon(u_\mu)$-term leads to

$$\int_\Omega D_t^{-k}\varepsilon(\dot u_\mu) : D_t^{-k}\tau \mathrm{d}x = \int_\Omega AD_t^{-k}\dot\sigma_\mu : D_t^{-k}\sigma_\mu + HD_t^{-k}\dot\xi_\mu : D_t^{-k}\xi_\mu \mathrm{d}x + \underbrace{\int_\Omega D_t^{-k}G'_\mu(\sigma_\mu,\xi_\mu) : D_t^{-k}(\sigma_\mu,\xi_\mu)\mathrm{d}x\,.}_{\ge 0} \tag{8.7}$$

By virtue of the monotonicity of $G'_\mu$, the tested penalty term is non negative. We now integrate (8.7) from $t_0$ to $T-\rho$ and use discrete partial integration on the left handside

$$\begin{aligned} -\int_{t_0+k}^{T-\rho}\int_\Omega \varepsilon(\dot u_\mu) : D_t^{+k}D_t^{-k}\tau \mathrm{d}x\mathrm{d}t + \frac{1}{k}\int_{t_0}^{t_0+k}\int_\Omega \varepsilon(\dot u_\mu)(t) : D_t^{-k}\tau \mathrm{d}x\mathrm{d}t - \frac{1}{k}\int_{T-\rho}^{T-\rho+k}\int_\Omega \varepsilon(\dot u_\mu) : D_t^{-k}\tau \mathrm{d}x\mathrm{d}t \\ \ge \frac{1}{2}\int_\Omega AD_t^{-k}\sigma_\mu : D_t^{-k}\sigma_\mu + HD_t^{-k}\xi_\mu : D_t^{-k}\xi_\mu \mathrm{d}x\Big|_{t_0}^{T}\,. \end{aligned} \tag{8.8}$$

We now let $\rho \to 0$, by a theorem about Lebesgue points we procure from (8.8) for almost all $t_0 \in (0,T)$

$$-\int_{t_0}^T\int_\Omega \varepsilon(\dot u_\mu) : \ddot\tau \mathrm{d}x\mathrm{d}t + \int_\Omega \varepsilon(\dot u_\mu) : \dot\tau \mathrm{d}x\big|_{t_0} - \int_\Omega \varepsilon(\dot u_\mu) : \dot\tau \mathrm{d}x\big|_T \ge \frac{1}{2}\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\Big|_{t_0}^{T}\,. \tag{8.9}$$

Integrating this inequality over $t_0$ form 0 to $h$ and multiplying with $\frac{1}{h}$ yields

$$\frac{1}{2}\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\big|_T \le \frac{1}{2}\frac{1}{h}\int_0^h\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\mathrm{dt} + \underbrace{\frac{1}{h}\int_0^h\int_\Omega \varepsilon(\dot u_\mu) : \dot\tau \mathrm{d}x\mathrm{dt}}_{(*)}$$
$$+ \underbrace{\int_\Omega \varepsilon(\dot u_\mu) : \dot\tau \mathrm{d}x\big|_T}_{(**)} + \hat K \,. \tag{8.10}$$

The constant $\hat K$ comes from the term $\int_{t_0}^T\int_\Omega \varepsilon(\dot u_\mu) : \ddot\tau \mathrm{d}x\mathrm{dt}$:
We used Young's inequality then the extended safe load condition (8.1) and the uniform $L^2(L^2)$-estimates from section 7. The terms $(*), (**)$ can be absorbed, to do this we first use Young's inequality and then theorem 8.2 . Thus

$$\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\big|_T \le \frac{1}{h}\int_0^h\int_\Omega A\dot\sigma_\mu : \dot\sigma_\mu + H\dot\xi_\mu : \dot\xi_\mu \mathrm{d}x\mathrm{dt} + K\,. \tag{8.11}$$

And this is the statement of the theorem. □

**Theorem 8.4** *We have* $(\dot\sigma_\mu, \dot\xi_\mu) \in L^\infty(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{sym})) \times L^\infty(0,T; L^2(\Omega, \mathbb{R}^m))$, *where* $m = 1$ *or* $m = n \times n$ *. Furthermore* $(\dot\sigma_\mu, \dot\xi_\mu)(0)$ *exists.*

**proof** This follows by the preceeding theorems of this section. □

This theorem and the convergence of $(\dot\sigma_\mu, \dot\xi_\mu) \to (\sigma, \xi)$ finaly gives:

**Theorem 8.5** *The time derivatives* $(\dot\sigma_\mu, \dot\xi_\mu)$ *of the stress tensor and hardening parameter, solution of* (1.2) *belong to the space* $L^\infty(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{sym})) \times L^\infty(0,T; L^2(\Omega, \mathbb{R}^m))$, *where* $m = 1$ *or* $m = n \times n$ *and* $(\dot\sigma, \dot\xi)(0)$ *exists.*

As a consequence we note

**Theorem 8.6** *The strain tensor* $\varepsilon(\dot u_\mu)$ *of the displacement velocities is contained in the space* $L^\infty(0,T; L^2(\Omega, \mathbb{R}^{n\times n}_{sym}))$, *with the uniform estimate*

$$\|\varepsilon(\dot u_\mu)\|_{L^\infty(L^2)} \le Const$$

*as* $\mu \to 0$.

## 9 A static hardening model

In the previous sections we considered a quasi-static hardening problem, whose formulation is due to Johnson [Joh78]. Following Temam [Tem85] we introduce a static hardening model based on the Hencky model of perfect elasto-plasticity.

We define the following sets:

$$\mathcal{K} = \{(\sigma,\xi)\,|\,F(\sigma,\xi) \le 0\}$$

$$\mathcal{M} = \{(\sigma,\xi) \in L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times L^2(\Omega,\mathbb{R}^m)\,|\,\sigma\cdot\vec{n} = p,\ \operatorname{div}\sigma \in L^n(\Omega,\mathbb{R}^n)\}\,.$$

Then the static hardening Problem is to find a minimum of

$$J(\sigma,\xi) = \frac{1}{2}\big(A\sigma,\sigma\big) + \frac{1}{2}\big(H\xi,\xi\big) \tag{9.1}$$

over the set $\mathcal{K}\cap\mathcal{M}\cap\{\sigma| -\operatorname{div}\sigma = f \text{ in } \Omega\}$. Where

$$f \in L^n(\Omega,\mathbb{R}^n)$$
$$p \in L^\infty(\partial\Omega,\mathbb{R}^n)\,.$$

The weak formulation is, to find $\big((\sigma,\xi),u\big) \in (\mathcal{M}\cap\mathcal{K})\times BD(\Omega)$ such that

$$\big(A\sigma,\tau-\sigma\big) + \big(H\xi,\eta-\xi\big) + \langle u,\operatorname{div}(\tau-\sigma)\rangle \ge 0 \tag{9.2}$$

$$\begin{aligned}\big(\sigma,\nabla w\big) &= \langle f,w\rangle + \int_{\Gamma_N} pw\,do \quad \forall w \in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n)\\ u &= 0 \text{ on } \Gamma_D\,.\end{aligned} \tag{9.3}$$

We will use the same type of approximation as in section 2, and we obtain similar results.

The penalized problem is to find $((\sigma_\mu,\xi_\mu),u_\mu) \in \mathcal{M}\times BD(\Omega)$ such that

$$\big(A\sigma,\tau-\sigma\big) + \big(H\xi,\eta-\xi\big) + \langle u,\operatorname{div}(\tau-\sigma)\rangle + \big(G'_\mu((\sigma_\mu,\xi_\mu)),(\tau-\sigma_\mu,\eta-\xi)\big) = 0 \tag{9.4}$$

$$\begin{aligned}\big(\sigma_\mu,\nabla w\big) &= \langle f,w\rangle + \int_{\Gamma_N} pw\,do \quad \forall w \in H^1_{\Gamma_D}(\Omega,\mathbb{R}^n)\\ u_\mu &= 0 \text{ on } \Gamma_D\,.\end{aligned} \tag{9.5}$$

Under the assumption of a safe load condition we are able the show the existence of the solutions to the static and penalized static hardening problem. We have the estimates for the solutions of the penalized problem independent of $\mu$

$$\|\sigma_\mu\|_{L^2} \le Const$$
$$\|\xi_\mu\|_{L^2} \le Const\,.$$

Like in the case of quasi-static hardening, for fixed viscosity coefficient $\mu$ the displacement is contained in $H^1(\Omega, \mathbb{R}^n)$

$$\|u_\mu\|_{H^1} \le \frac{1}{\mu} Const$$
$$\|\varepsilon(u_\mu)\|_{L^2} \le \frac{1}{\mu} Const\,.$$

And the pointwise almost everywhere equation

$$\begin{pmatrix} A\sigma_\mu \\ H\xi_\mu \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial \sigma_\mu} G_\mu(\sigma_\mu, \xi_\mu) \\ \frac{\partial}{\partial \xi_\mu} G_\mu(\sigma_\mu, \xi_\mu) \end{pmatrix} = \begin{pmatrix} \varepsilon(u_\mu) \\ 0 \end{pmatrix} \tag{9.6}$$

holds in $\Omega$. The convergence of the solutions of the penalized model can be shown in the same way as in section 6.

If we use the same(but not time dependent) testfunctions as in section 7, we obtain the uniform estimate $\|u_\mu\|_{H^1} \le Const$ and thus $u \in H^1$.
By using finite differencens, we are able to show that $\sigma, \xi \in H^1_{loc}$.

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