In class, I made such a large number of typos in this proof that it is probably unreadable in your notes. Here is a (hopefully) typo free version. Thank you to the students who pointed these typos.

We suppose that $d\mathcal{A}_H$ vanishes at a point $(\gamma, u) \in \widetilde{\mathcal{L}_0 M}$. Our goal is to prove that γ must be an orbit of H.

To this end, let $Y = (Y_t)$ be a vector field along γ, i.e. a tangent vector at the point $(\gamma, u) \in \widetilde{\mathcal{L}_0 M}$. Extend Y to a vector field V along u (meaning that $V(e^{2\pi i t}/2\pi) = Y_t$). Let D^2 be the disk of radius $1/2\pi$. Let $(u_s)_{s\in(-\epsilon,\epsilon)} : D^2 \to M$ be a family of maps with $\frac{d}{ds}|_{s=0} u_s = V$, $u_0 = u$.[1]

We compute

$$\begin{aligned}
d\mathcal{A}_H(Y) &= \frac{d}{ds}|_{s=0}\mathcal{A}(u_s) \\
&= \frac{d}{ds}|_{s=0}\left(-\int_{D^2} u_s^*\omega + \int_0^1 H_t u_s(e^{2\pi i t}/2\pi)\right) \\
&= -\int_{D^2} u^*\mathcal{L}_V\omega + \int_0^1 dH_t(\partial_s u_s(e^{2\pi i t}/2\pi)|_{s=0}) \\
&= -\int_{D^2} u^*(d i_V \omega) + \int_0^1 dH_t(Y_t) \\
&= -\int_0^1 \omega(Y_t, \dot\gamma(t)) - \int_0^1 \omega(X_t^H(\gamma(t)), Y_t) \\
&= \int_0^1 \omega(\dot\gamma(t) - X_t^H(\gamma(t)), Y_t)
\end{aligned}$$

Since Y was arbitrary, we conclude that

$$\dot\gamma(t) = X_t^H(\gamma(t)).$$

[1] In class, I constructed this family using the exponential map. I also called it v_s instead of u_s, which caused understandable confusion with the vector field V....