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In this note I review three fields found in Conway's group of games: the surreal numbers, the real numbers, and the nimbers. The primary motivation to write this note was to understand why the multiplication of surreal numbers is well-defined; most proofs of this fact in the literature seem to follow Conway's original book [Con01], but the argument given there does not quite follow the inductive scheme that it claims to follow, so substantial reorganization is required. After finding one such reorganization I became aware that a better one has been already worked out by Schleicher and Stoll [SS06]. Hence the main original aspect of the present write-up is that I make the symmetry between Left and Right options of a game in multiplicative constructions very explicit. This idea arose in a discussion with Stefan Rabenstein.

1 Games

Definition 1.1. A *game* consists of its left options and right options. All games are constructed in this way.

A generic left (resp. right) option of x is denoted by x_L (resp. x_R). In order to highlight the symmetry between Left and Right we use placeholders $j,\alpha,\beta,\gamma,\dots$ for L,R. Most statements are prepended with quantifiers $\forall j,\alpha,\beta,\gamma,\dots$ unless mentioned otherwise. It is convenient to identify $L=+1, R=-1$. We write $x_j \in_j x$.

We define a transitive relation $\prec$ on games by $x \prec y$ iff there is a chain $x \in_{\alpha_1} x_1 \in_{\alpha_2} \cdots \in_{\alpha_n} y$. The statement "all games are constructed in this way" is formalized by demanding that $\prec$ is well-founded. Theorems about games are proved by induction on $\prec$ and theorems about pairs of games by induction on the product order unless mentioned otherwise.

Definition 1.2.

$$(x+y)_\alpha :\equiv x_\alpha + y, x + y_\alpha.$$

$$(-x)_\alpha :\equiv -(x_{-\alpha})$$

Observation: addition is commutative and distributive. Write $x-y :\equiv x+(-y)$.

Definition 1.3. The game without options is denoted by 0. Recursively define

$$x \gtreqless_j 0 :\iff \forall x_{-j}\ (x_{-j} \not\gtreqless_{-j} 0)$$

The relation $x \gtreqless_j 0$ is interpreted as "j wins x when moving second". Indeed, by the recursive definition this happens exactly when all first moves of the opponent $-j$ lead to a game which $-j$ cannot win when j is on turn.

Lemma 1.4.

$$x \gtreqless_j 0 \iff -x \gtreqless_{-j} 0.$$

Proof. By induction we may assume that the conclusion is known for all options of x. Then

$$-x \gtreqless_j 0 \iff \forall(-x)_{-j}\ ((-x)_{-j} \not\gtreqless_{-j} 0) \iff \forall x_j\ (-(x_j) \not\gtreqless_{-j} 0)$$
$$\iff \forall x_j\ (x_j \not\gtreqless_j 0) \iff x \gtreqless_{-j} 0.$$

□

Lemma 1.5. *Suppose* $x \gtreqless_j 0$ *and* $y \gtreqless_j 0$*. Then* $x+y \gtreqless_j 0$.

The strategy for playing $x+y$ is to answer all moves of the opponent in the same game.

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Proof. We induct on x, y.

We have to show $(x+y)_{-j} \not\lesseqgtr_{-j} 0$ for every $-j$-th option of $x+y$. By symmetry it suffices to consider $x_{-j}+y$. By the assumption $x \lesseqgtr_j 0$ we have $x_{-j} \not\lesseqgtr_{-j} 0$, so there is an option with $(x_{-j})_j \lesseqgtr_j 0$. Hence by the inductive hypothesis $(x_{-j})_j + y \lesseqgtr_j 0$. This is a j-th option of $x_{-j}+y$ and we conclude by definition. □

Lemma 1.6. *For any game x and any j we have $x - x \lesseqgtr_j 0$.*

Here we use the symmetric play strategy.

Proof. We induct on x.

We have to show $(x-x)_{-j} \not\lesseqgtr_{-j} 0$ for every choice of the option on the right-hand side. This option can take two possible forms. The first case is $x_{-j} - x$. Here we use the inductive hypothesis to conclude $x_{-j} - x_{-j} \lesseqgtr_j 0$ and plug this into the definition of $\lesseqgtr_{-j}$ on $x_{-j} - x$. The second case $x - x_j$ is similar. □

Corollary 1.7. *For every game x and $j = \pm 1$ we have*

$$x_j \not\lesseqgtr_j x.$$

Proof. Equivalently, there is an option z_{-j} of $z :\equiv x_j - x$ with $z_{-j} \lesseqgtr_{-j} 0$. One such option is $x_j - x_j$. □

Lemma 1.8. *For any games x, y we have $x + y \lesseqgtr_j 0 \wedge y \lesseqgtr_{-j} 0 \implies x \lesseqgtr_j 0$.*

If $x \not\lesseqgtr_j 0$, then $-j$ wins x as the first player. $-j$'s strategy for playing $x+y$ is then to move in x first.

Proof. Suppose $x \not\lesseqgtr_j 0$. Then there is an option $x_{-j} \lesseqgtr_{-j} 0$. By Lemma 1.5 we have $x_{-j}+y \lesseqgtr_{-j} 0$. But the left hand side is an option of $x+y$, and by definition we obtain $x+y \not\lesseqgtr_j 0$, a contradiction. □

We extend $\lesseqgtr_j$ to binary relations on games by $x \lesseqgtr_j y :\iff (x-y) \lesseqgtr_j 0$.

Corollary 1.9. *The relations $\lesseqgtr_j$ are reflexive, transitive, and translation invariant in the sense $x \lesseqgtr_j y \iff (x+z) \lesseqgtr_j (y+z)$.*

Proof. Reflexivity is given by Lemma 1.6.

To show transitivity suppose $x \lesseqgtr_j y \lesseqgtr_j z$. By definition this means $(x-y) \lesseqgtr_j 0$ and $(y-z) \lesseqgtr_j 0$. By Lemma 1.5 this implies $(x-y+y-z) \lesseqgtr_j 0$. By Lemma 1.6 we have $-y+y \lesseqgtr_{-j} 0$. By Lemma 1.8 this implies $(x-y) \lesseqgtr_j 0$.

Translation invariance follows from $z \lesseqgtr_{\pm j} z$ and Lemmas 1.5 and 1.9 for the forward and the backward implication, respectively. □

The relations $\lesseqgtr_1$ and $\lesseqgtr_{-1}$ are usually called $\geq$ and $\leq$, respectively, but the author could not resist indexing them. Equality is defined by

$$x = y :\iff x \leq y \wedge x \geq y.$$

Finally, we write

$$x \lessgtr_j y :\iff x \lesseqgtr_j y \wedge x \not\lesseqgtr_{-j} y.$$

Corollary 1.9 shows that games form a partially ordered Group modulo $=$.

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Lemma 1.10. *The relations* $\lessgtr_j$ *are transitive.*

Proof. Suppose $x \lessgtr_j y \lessgtr_j z$. By transitivity of $\lesseqgtr_j$ we obtain $x \lesseqgtr_j z$. Suppose now $x \lesseqgtr_{-j} z$. Then also $z \lesseqgtr_j x$, so by transitivity of $\lesseqgtr_j$ we obtain $y \lesseqgtr_j x$, a contradiction. □

Lemma 1.11 (One-sided Simplest Form Theorem for games)**.** *Let* x,z *be games and* j *be given. If* $z \not\lesseqgtr_{-j} x_j$ *for all options of* x *and for every option* z_{-j} *there is an option* x_{-j} *with* $z_{-j} \lesseqgtr_j x_{-j}$*, then* $z \lesseqgtr_j x$*.*

Proof. We have to show $z_{-j} \not\lesseqgtr_{-j} x$ and $z \not\lesseqgtr_{-j} x_j$ for all respective options. The second relation is just the hypothesis. On the other hand, by the hypothesis we have $z_{-j} \lesseqgtr_j x_{-j}$ for some option of x. By definition this implies $z_{-j} \not\lesseqgtr_{-j} x$ as required. □

Lemma 1.12. *Let* x, y *be games and let* $j \in \{\pm 1\}$ *be such that* $\forall y_j \exists x_j : x_j \lesseqgtr_j y_j$ *and* $\forall x_{-j} \exists y_{-j} : x_{-j} \lesseqgtr_j y_{-j}$*. Then* $x \lesseqgtr_j y$*.*

Proof. The first condition implies $x \not\lesseqgtr_{-j} y_j$ for all y_j. The conclusion follows from Lemma 1.11. □

Applying the last Lemma with both $j = \pm 1$ we obtain that $x = y$ follows from analogous relations for the respective options. This will be used repeatedly in the proofs of the distributive and associative laws for multiplication.

Definition 1.13. For games x, y define

$$(xy)_{\alpha\beta} :\equiv x_\alpha y + x y_\beta - x_\alpha y_\beta.$$

Lemma 1.14. *For any games* x, y, z *we have*

1. $x0 \equiv 0$*, where* $0 :\equiv \{|\}$*,*
2. $x1 \equiv x$*, where* $1 :\equiv \{0|\}$*,*
3. $xy \equiv yx$
4. $(-x)y \equiv -(xy) \equiv x(-y)$
5. $(x+y)z = xz + yz$
6. $(xy)z = x(yz)$

Proof. In the product 1 there are no options. In the product 2 the options are

$$(x1)_{\alpha 1} \equiv x_\alpha 1 + x0 + x_\alpha 0 \equiv x_\alpha 1 \equiv x_\alpha$$

by induction and part 1. Commutativity 3 is proved by induction using commutativity of addition:

$$(xy)_{\alpha\beta} \equiv x_\alpha y + x y_\beta - x_\alpha y_\beta \equiv y x_\alpha + y_\beta x - y_\beta x_\alpha \equiv y_\beta x + y x_\alpha - y_\beta x_\alpha \equiv (yx)_{\beta\alpha}.$$

In 4 we only need to show the first identity since the second follows from it by commutativity. By induction we have

$$\begin{aligned}((-x)y)_{\alpha\beta} &\equiv (-x)_\alpha y + (-x)y_\beta - (-x)_\alpha y_\beta \equiv (-(x_{-\alpha}))y + (-x)y_\beta - (-(x_{-\alpha}))y_\beta \\ &\equiv -x_{-\alpha} y - x y_\beta + x_{-\alpha} y_\beta \equiv -(x_{-\alpha} y + x y_\beta - x_{-\alpha} y_\beta) \equiv -(xy)_{-\alpha\beta} \equiv (-(xy))_{\alpha\beta}.\end{aligned}$$

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To see distributivity 5 write without loss of generality

$$((x+y)z)_{\alpha\beta} \equiv (x+y)_\alpha z + (x+y)z_\beta - (x+y)_\alpha z_\beta \equiv (x_\alpha + y)z + (x+y)z_\beta - (x_\alpha + y)z_\beta$$

and using the inductive hypothesis and Lemma 1.6 this is

$$= x_\alpha z + yz + xz_\beta + yz_\beta - x_\alpha z_\beta - yz_\beta \equiv (xz)_{\alpha\beta} + yz + yz_\beta - yz_\beta = (xz)_{\alpha\beta} + yz \equiv (xz+yz)_{\alpha\beta}.$$

The claimed equality follows by Lemma 1.12. To show the equality 6 expand the options on the left-hand side as

$$((xy)z)_{\alpha\beta\gamma} \equiv (xy)_{\alpha\beta} z + (xy)z_\gamma - (xy)_{\alpha\beta} z_\gamma \equiv (x_\alpha y + xy_\beta - x_\alpha y_\beta)z + (xy)z_\gamma - (x_\alpha y + xy_\beta - x_\alpha y_\beta)z_\gamma.$$

Using the inductive hypothesis and distributivity (part 5) this is

$$= x_\alpha(yz) + x(y_\beta z + yz_\gamma - y_\beta z_\gamma) - x_\alpha(y_\beta z + yz_\gamma - y_\beta z_\gamma) \equiv (x(yz))_{\alpha\beta\gamma}.$$

The claimed equality again follows by Lemma 1.12. □

2 Numbers

Definition 2.1. A game x is called a *number* if for all options we have $x_j \not\overset{\leq}{>}_j x_{-j}$ and every option is also a number.

Proposition 2.2. *For every number x we have $x \lessgtr_j x_j$.*

Proof. In view of Corollary 1.7 it suffices to show $x_j - x \overset{\leq}{>}_{-j} 0$. Equivalently, for all z_j, $z :\equiv x_j - x$, we have $z_j \not\overset{\leq}{>}_j 0$. We have two possibilities for z_j. One is given by $x_j - x_{-j}$, and here we use the assumption that x is a number. The other is given by $(x_j)_j - x$. Suppose for contradiction $(x_j)_j \overset{\leq}{>}_j x$. By induction hypothesis we also have $x_j \lessgtr_j (x_j)_j$, so by transitivity (Corollary 1.9) we get $x_j \overset{\leq}{>}_j x$, contradicting Corollary 1.9. □

Lemma 2.3. *Numbers are a subgroup of games.*

Proof. The hardest part is to show that the numbers are closed under addition. If x, y are numbers then wlog the options of $z :\equiv x + y$ are $x_j + y$ and by Proposition 2.2 and Corollary 1.9 we have $z = x + y \lessgtr_j x_j + y = z_j$. We conclude using transitivity of $\lessgtr_j$. □

Corollary 2.4. *Numbers are totally ordered by $\leq$.*

Proof. By Lemma 2.3 it suffices to show that numbers are comparable to 0. Fix $j \in \{\pm 1\}$. If we have $x_j \overset{\leq}{>}_j 0$ for some option then by Proposition 2.2 and transitivity (Corollary 1.9) we obtain $x \overset{\leq}{>}_j 0$. Otherwise by definition we have $x \overset{\leq}{>}_{-j} 0$. □

Theorem 2.5 (Simplest Form). *Let x be a game and z a number. If $z \not\overset{\leq}{>}_j x_{-j}$ for all options of x, but no option of z satisfies this condition, then $z = x$.*

Proof. Suppose $z_j \overset{\leq}{>}_{-\alpha} x_\alpha$. If $\alpha = -j$, then by Proposition 2.2 we get $z \lessgtr_j z_j \overset{\leq}{>}_j x_{-j}$, so by transitivity $z \overset{\leq}{>}_j x_{-j}$, contradicting the hypothesis. So we must have $\alpha = j$. Hence x, z satisfy the assumptions of Lemma 1.11 for $j = \pm 1$. □

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2.1 Multiplication

Theorem 2.6. *Let x, y, y' be numbers. Then*

1. *xy is a number.*
2. *For every option x_α of x we have*
$$(y - y') \lesseqgtr_j 0 \implies (x - x_\alpha)(y - y') \lesseqgtr_{\alpha j} 0$$
3. *We have*
$$(y - y') = 0 \implies x(y - y') = 0$$

Claim 3 in particular shows that multiplication is well-defined modulo $=$.

Proof. The theorem is proved by induction on a relation $\prec'$ on tuples $(x, \{y, y'\})$ consisting of a number and an unordered pair of numbers. The relation $\prec'$ is the smallest transitive relation for which the following elements precede $(x, \{y, y'\})$:

$$\begin{aligned}
&(\tilde{x}, \{y, y'\}), && \tilde{x} \prec x\\
&(x, \{\tilde{y}, y'\}), && \tilde{y} \prec y\\
&(x, \{\tilde{y}, \tilde{y}'\}), && \tilde{y} \prec y, \tilde{y}' \prec y\\
&(y, \{\tilde{x}, \tilde{x}'\}), && \tilde{x} \prec x, \tilde{x}' \prec x.
\end{aligned}$$

Using the axiom of choice it is not hard to verify that the relation $\prec'$ is in fact transitive. Question: do we need choice here?

Proof of claim 1 By Claim 1 in the inductive hypothesis and Lemma 2.3 all options of xy are numbers. It remains to verify $(xy)_1 \not\geq (xy)_{-1}$ or, equivalently,

$$-(x - x_\alpha)(y - y_\beta) < -(x - x_\gamma)(y - y_\delta)$$

provided $\alpha\beta = 1$, $\gamma\delta = -1$. This is equivalent to

$$(x - x_\alpha)(y - y_\beta) - (x - x_\gamma)(y - y_\delta) > 0.$$

This is symmetric in simultaneously permuting γ, δ and x, y, so assume $\gamma = 1$, $\delta = -1$. By Lemma 1.6 and Lemma 1.14 the left-hand side is

$$= (x_\gamma - x_\alpha)(y - y_\beta) + (x - x_\gamma)(-y_\beta + y_\delta) = (x - x_\alpha)(y_\delta - y_\beta) + (-x_\alpha + x_\gamma)(y - y_\delta).$$

If $\alpha = \beta = 1$ then both products involving $x - x_1$ are > 0 and at least one of the products involving $(x_\gamma - x_\alpha)$ is ≥ 0 by Claims 2 and 3 in the inductive hypothesis. If $\alpha = \beta = -1$ then similarly both products involving $y - y_{-1}$ are > 0 and one of the other products is ≥ 0. In any case we have obtained at least one expression that is > 0.

Proof of claim 2 In case $y \lesseqgtr_j y'$ we have $y \lesseqqgtr_j \tilde{y} \lesseqqgtr_j y'$ for $\tilde{y}$ an option of either y or y' and we split
$$(x - x_\alpha)(y - y') = (x - x_\alpha)(y - \tilde{y}) + (x - x_\alpha)(\tilde{y} - y').$$

One of the products (in which $\tilde{y}$ replaces its parent) is $\lesseqqgtr_{\alpha j} 0$ by Claims 2 and 3 in the inductive hypothesis. The other product (in which $\tilde{y}$ figures along with its parent) is $\lesseqgtr_{\alpha j} 0$ since xy, xy' are numbers, see Claim 1.

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Proof of claim 3 The options of $x(y-y')$ are

$$x_\alpha(y-y')+x(y-y')_\beta-x_\alpha(y-y')_\beta. \tag{2.7}$$

By the Simplest Form Theorem 2.5 it suffices to show that these options are $\lesseqgtr_{-\alpha\beta} 0$. The first term is $=0$ by Claim 3 in the inductive hypothesis. The second and the third term combine to

$$=(x-x_\alpha)(y-y')_\beta,$$

and the latter bracket has the form $(y-y')_\beta \equiv \tilde{y}-\tilde{y}' \lesseqgtr_{-\beta} y-y'=0$ with a pair $(\tilde{y},\tilde{y}')$ strictly preceding (y,y'). Hence the product is $\lesseqgtr_{-\alpha\beta} 0$ by Claim 2 in the inductive hypothesis. □

Corollary 2.8. *For any numbers x,y we have $x>0 \wedge y>0 \implies xy>0$.*

Proof. Define a number x' by $0 \in_1 x'$ and $x_\alpha \in_\alpha x'$. Then $x=x'$ by the Simplest Form Theorem 2.5, so $xy=x'y$. Now $0\in_1 x'$, so $x'y>0$ follows from Claim 2 in Theorem 2.6. □

Thus numbers modulo $=$ form a totally ordered Integral Domain.

2.2 Multiplicative inverse

Definition 2.9. Let $x>0$ be a number. We define $0\in_1 x^{-1}$ and

$$(x^{-1})_{-\alpha\beta} :\equiv (1+(x_\alpha-x)(x^{-1})_\beta)(x_\alpha)^{-1}, \quad x_\alpha>0.$$

Lemma 2.10. *1.* $1-x(x^{-1})_\alpha \lesseqgtr_\alpha 0$.

2. x^{-1} *is a number.*

3. $x(x^{-1})=1$.

The lemma shows that non-zero Numbers have multiplicative inverses. These inverses are well-defined modulo $=$ because Numbers are an Integral Domain modulo $=$.

Proof. We induct on x.

Proof of claim 1 For the option 0 this is immediate. For the other options we have

$$\begin{aligned}1-x(x^{-1})_{-\alpha\beta} &\equiv 1-x(1+(x_\alpha-x)(x^{-1})_\beta)(x_\alpha)^{-1}\\ &=(x_\alpha-x)(1-x(x^{-1})_\beta)(x_\alpha)^{-1}\\ &\lesseqgtr_{-\alpha\beta} 0,\end{aligned}$$

where we have used the fact that Numbers form an Integral Domain and the inductive hypothesis.

Claim 2 follows from claim 1 and the fact that multiplication by the positive number x is an order-preserving operation on Numbers.

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Proof of claim 3 For the options of $x(x^{-1})$ we have

$$(x(x^{-1}))_{\alpha\beta} - 1 \equiv x_\alpha(x^{-1}) + x(x^{-1})_\beta - x_\alpha(x^{-1})_\beta - 1$$

If $x_\alpha > 0$ then we write this as

$$\begin{aligned}
&= x_\alpha(x^{-1}) - (1 + (x_\alpha - x)(x^{-1})_\beta) \\
&= x_\alpha(x^{-1}) - x_\alpha(1 + (x_\alpha - x)(x^{-1})_\beta)(x_\alpha)^{-1} \\
&= x_\alpha(x^{-1}) - x_\alpha(x^{-1})_{-\alpha\beta} \\
&= x_\alpha(x^{-1} - (x^{-1})_{-\alpha\beta}) \\
&\lesseqgtr_{-\alpha\beta} 0.
\end{aligned}$$

If $x_\alpha \leq 0$ then we write the above as

$$= x_\alpha((x^{-1}) - (x^{-1})_\beta) - (1 - x(x^{-1})_\beta).$$

Both brackets are $\lesseqgtr_\beta 0$ by claims 2 and 1, respectively, so by monotonicity of multiplication we obtain that the expression is $\lesseqgtr_{-\beta} 0$. Since in this case necessarily $\alpha = 1$ we obtain altogether

$$(x(x^{-1}))_{\alpha\beta} - 1 \lesseqgtr_{-\alpha\beta} 0.$$

The Simplest Form Theorem 2.5 now implies either $x(x^{-1}) = 1$ or $x(x^{-1}) = 0$. But in the latter case we would have $x^{-1} = 0$, since Numbers are an Integral Domain, which contradicts the fact that 0 is by definition an option of x^{-1}. □

3 Dyadic numbers

Since numbers form a totally ordered Field, they contain a unique copy of the dyadic rationals. As a ring this copy is generated by $\frac{1}{2} :\equiv \{0|1\}$ (we have $\frac{1}{2} + \frac{1}{2} \equiv \{0 + \frac{1}{2}|1 + \frac{1}{2}\} = 1$ by the Simplest Form Theorem 2.5, so this notation makes sense).

Lemma 3.1. *For every $m \in \mathbb{Z}$ we have $m \lesseqgtr_j 0 \implies m = z$, where z is the number with the only option $m - j :\in_j z$. We also have $\frac{m}{2^n} = \{\frac{m-1}{2^n}|\frac{m+1}{2^n}\}$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.*

Proof. In the first claim we induct on m. For $m = 0$ this follows from the Simplest Form Theorem 2.5. For positive m we have

$$(m-1) + 1 = \{m-2|\} + \{0|\} = \{m-2, m-1|\} = \{m-1|\}$$

by the Simplest Form Theorem 2.5. Negative m are handled analogously.

The second claim for $n = 0$ and arbitrary m follows from the first claim by the Simplest Form Theorem 2.5. Suppose now that it is valid for some n. Let $z :\equiv \{\frac{m-1}{2^{n+1}}|\frac{m+1}{2^{n+1}}\}$. Then

$$z + z \equiv \{\frac{m-1}{2^{n+1}} + z|\frac{m+1}{2^{n+1}} + z\}.$$

Since $\frac{m-1}{2^{n+1}} < z < \frac{m+1}{2^{n+1}}$, by the Simplest Form Theorem 2.5, and by the inductive hypothesis we obtain

$$z + z = \{\frac{m-1}{2^{n+1}} + \frac{m-1}{2^{n+1}}|\frac{m+1}{2^{n+1}} + \frac{m+1}{2^{n+1}}\} = \{\frac{m-1}{2^n}|\frac{m+1}{2^n}\} = \frac{m}{2^n},$$

so $z = \frac{m}{2^{n+1}}$ as required. □

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Theorem 3.2. *Let $\mathbb{D} \subset \mathrm{No}$ be the minimal subring with $L, R \subset \mathbb{D}$ finite, $L < R \implies \{L|R\} \in \mathbb{D}$. Then $\mathbb{D}$ is generated by $\frac{1}{2}$ modulo $=$.*

Proof. Note that by the Simplest Form Theorem 2.5 it suffices to consider one-element subsets of $\mathbb{D}$ in the above recursive definition of $\mathbb{D}$. Thus it suffices to show that for any dyadic numbers $\frac{m}{2^n}, \frac{m'}{2^n}$, $m < m'$ the number $z :\equiv \{\frac{m}{2^n}|\frac{m'}{2^n}\}$ is again dyadic. If $z = \frac{m''}{2^n}$ for some $m < m'' < m'$, then we are done. Otherwise by the pigeonhole principle we have

$$\frac{k}{2^n} < z < \frac{k+1}{2^n}$$

for some $m \le k < m'$. It follows from the Simplest Form Theorem 2.5 that

$$z = \{\frac{k}{2^n}|\frac{k+1}{2^n}\},$$

and the latter is $= \frac{2k+1}{2^{n+1}}$ by Lemma 3.1. □

4 Real numbers

Let $\mathbb{D} \subset \mathrm{No}$ be a subring with $1 \in \mathbb{D}$.

Definition 4.1. The set of strictly positive numbers in $\mathbb{D}$ is denoted by $\mathbb{D}_+$. The *standard part* of a number x is the number with the options

$$\mathrm{st}(x)_\alpha = x - \alpha q, \quad q \in \mathbb{D}_+.$$

A number x is called *infinitesimal* ($x \in \mathbb{I}$) if

$$\forall q \in \mathbb{D}_+ : -q < x < q.$$

A number x is called *limited* ($x \in \mathbb{L}$) if

$$\exists q \in \mathbb{D}_+ : -q < x < q.$$

It is easy to see that the standard part of a number is indeed a number.

We make the qualitative assumption $\forall q \in \mathbb{D}, q > 0 \exists r, r' \in \mathbb{D} : 0 < r < q^{-1} < r'$. The simplest example is the set of numbers with finite birthday (equivalently, the subring generated by $\frac{1}{2}$; the latter equivalence is not entirely trivial).

Lemma 4.2. $\mathbb{L} \subset \mathrm{No}$ *is a subring.* $\mathbb{I} \subset \mathbb{L}$ *is an ideal.*

Proof. The hardest part is to show $xy \in \mathbb{I}$ provided $x \in \mathbb{I}$, $y \in \mathbb{L}$. Without loss of generality $x, y > 0$. We have $y < Q \in \mathbb{D}_+$ and there exists $r \in \mathbb{D}_+$ with $r < Q^{-1}$. Let $q \in \mathbb{D}_+$. Then $x < qr < qQ^{-1}$, so $xy < xQ < q$ as required. □

Lemma 4.3. *For every number x the difference $\mathrm{st}\, x - x$ is infinitesimal.*

Proof. The difference $\mathrm{st}\, x - x$ is a number and its options include

$$(\mathrm{st}\, x - x)_\alpha = \mathrm{st}(x)_\alpha - x = (x - \alpha q) - x = -\alpha q.$$

In particular, the left options contain $-\mathbb{D}_+$ and the right options contain $\mathbb{D}_+$. □

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Lemma 4.4. *Suppose that the options of z differ from the options of* $\mathrm{st}(x)$ *at most by infinitesimal numbers. Then* $z = \mathrm{st}(x)$.

Proof. The right options of x correspond to $q \in \mathbb{D}_+$. For a given $q \in \mathbb{D}_+$ take $r \in \mathbb{D}_+$ with $r < q$, then

$$x + q > x + r + \epsilon$$

for any $\epsilon \in \mathbb{I}$, and in particular $x + q$ is greater than a right option of z, so it is greater than z. The analogous argument for left options shows that z satisfies the condition in Theorem 2.5. Conversely, every option of z has the form $x - \alpha a$, $a \in \mathbb{D}_+ + \mathbb{I}$, so it is separated from x by an option of x and does not satisfy the condition in Theorem 2.5. □

Corollary 4.5. *For every number x we have* $\mathrm{st}(\mathrm{st}(x)) = \mathrm{st}(x)$.

Proof. On the left-hand side we have the options

$$(\mathrm{st}(\mathrm{st}(x)))_\alpha = \mathrm{st}(x) - \alpha q,$$

and by Lemma 4.3 they differ at most infinitesimally from the options on the right-hand side. Lemma 4.4 concludes the proof. □

Lemma 4.6. *The standard part function is an additive group homomorphism.*

Proof. We have the easy identity

$$\mathrm{st}(-x) = -(\mathrm{st}\, x),$$

and in particular $\mathrm{st}(0) = 0$. It remains to show

$$x + y + z = 0 \implies \mathrm{st}\, x + \mathrm{st}\, y + \mathrm{st}\, z = 0.$$

Up to permutation of x, y, z the options have the form

$$(\mathrm{st}\, x + \mathrm{st}\, y + \mathrm{st}\, z)_\alpha = x - \alpha q + \mathrm{st}\, y + \mathrm{st}\, z.$$

By Lemma 4.3 this is an element of

$$x + y + z - \alpha q + \mathbb{I} = -\alpha q + \mathbb{I}.$$

This implies $\mathrm{st}\, x + \mathrm{st}\, y + \mathrm{st}\, z = 0$ by Lemma 4.4. □

Corollary 4.7. $\ker \mathrm{st} = \mathbb{I}$.

Proof. Inclusion "$\supseteq$" is given by Lemma 4.4 and inclusion "$\subseteq$" by Lemma 4.3. □

Lemma 4.8. *The standard part function preserves $\mathbb{L}$ and is multiplicative on $\mathbb{L}$.*

Proof. $\mathrm{st}(\mathbb{L}) \subset \mathbb{L}$ is clear. Let $x, y \in \mathbb{L}$. Then by Lemma 4.3 we have

$$\begin{aligned}(\mathrm{st}(x)\,\mathrm{st}(y))_{\alpha\beta} &= (x - \alpha q)\,\mathrm{st}(y) + \mathrm{st}(x)(y - \beta r) - (x - \alpha q)(y - \beta r)\\ &\in (x - \alpha q)y + x(y - \beta r) - (x - \alpha q)(y - \beta r) + \mathbb{I}\\ &= xy - \alpha\beta q r + \mathbb{I}.\end{aligned}$$

Hence the options of $\mathrm{st}(x)\,\mathrm{st}(y)$ differ from the options of $\mathrm{st}(xy)$ at most by infinitesimals, and this implies equality by Lemma 4.4. □

Definition 4.9. Let $\mathbb{R} := \mathrm{fix}\, \mathrm{st} \cap \mathbb{L}$ (equivalently, $\mathbb{R} = \mathrm{st}(\mathbb{L})$).

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Proposition 4.10. $\mathbb{R}$ *is a subfield of* No.

Proof. Standard part preserves $\mathbb{L}$, fixes 1, and is a ring homomorphism on $\mathbb{L}$, so $\mathbb{R}$ is a subring. It remains to show that for each $x \in \mathbb{R} \setminus \{0\}$ we have $x^{-1} \in \mathbb{R}$ (inverse taken in No). Without loss of generality suppose $x > 0$. By Corollary 4.7 we have $x \notin \mathbb{I}$. In particular, $x \geq q$ for some $q \in \mathbb{D}_+$. Hence $x^{-1} \leq q^{-1} < r$ for some $r \in \mathbb{D}_+$, so that $x^{-1} \in \mathbb{L}$. By Lemma 4.8 we have

$$\mathrm{st}(x^{-1})x = \mathrm{st}(x^{-1})\,\mathrm{st}(x) = \mathrm{st}(x^{-1}x) = \mathrm{st}(1) = 1,$$

so $\mathrm{st}(x^{-1}) = x^{-1}$ as required. $\square$

Lemma 4.11. *Suppose that $\mathbb{D}$ has the Archimedean property and contains the dyadic numbers. Then $\mathbb{D}$ is order dense in $\mathbb{R}$.*

Proof. Let $x, y \in \mathbb{R}$ with $x > y$. Without loss of generality suppose also $y > 0$. By Lemma 4.7 there exists $q \in \mathbb{D}_+$ with $x - y > q$. By the Archimedian property we have $x - y > 2^{-m}$ for some $m \in \mathbb{N}$. Moreover, since $x \in \mathbb{L}$ and by the Archimedian property, we also have $x < N$ for some $N \in \mathbb{N}$. It follows that there is a number of the form $k/2^{m+1}$, $k \in \mathbb{N}$, $k \leq 2^{m+1}N$, between x and y. Moreover, this number is contained in $\mathbb{D}$. $\square$

In particular, under the hypothesis of this lemma $\mathbb{R}$ can be identified with a set of Dedekind cuts of $\mathbb{D}$, ensuring that $\mathbb{R}$ is a set (and not a proper class).

Lemma 4.12. $x \geq 0 \implies \mathrm{st}(x) \geq 0$.

Proof. Suppose $\mathrm{st}(x) \ngeq 0$. Then there is a right option $\mathrm{st}(x)_{-1} \leq 0$. By definition of the standard part we have $x + q \leq 0$ for some $q \in \mathbb{D}_+$, and in particular $x < 0$, a contradiction. $\square$

Proposition 4.13. *The field $\mathbb{R}$ is order complete.*

Proof. Let $L \subset \mathbb{R}$ be a bounded above set and let $R \subset \mathbb{R}$ be the set of all upper bounds for L. If L contains a maximum then there is nothing to show. Otherwise let $z := \{L|R\}$. Then z is a limited number and we have $l < z < r$ for all $l \in L, r \in R$. By Lemma 4.12 and Lemma 4.6 it follows that $l \leq \mathrm{st}\, z \leq r$, so $\mathrm{st}\, z$ is in fact the supremum of L. $\square$

5 Nimbers

Definition 5.1. In an *impartial* game the set of left options coincides with the set of right options and both sets consist of impartial games. All impartial games are constructed in this way.

The impartial games form a Subgroup of games. By induction every impartial game is its own additive inverse. Options of an impartial game x will be denoted by x'.

Lemma 5.2. *For every impartial game x exactly one of the following holds.*

1. *For all options we have $x' \neq 0$, and then $x = 0$, or*
2. *there is an option with $x' = 0$, and then $x \ngeq_j 0$, $j = \pm 1$, and in particular $x \neq 0$.*

In particular, any two impartial games are either equal or incomparable under $\leq$.

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Proof. We argue by induction on x. The two statements about the options are complementary, so it remains to verify the consequences about x. Recall that by definition

$$x \lesseqgtr_j 0 \iff \forall x'(x' \not\lesseqgtr_{-j} 0).$$

Hence if the second option holds, then $x \not\lesseqgtr_j 0$ for $j = \pm 1$ as claimed. On the other hand, if the first option holds for x, then for each option x' the second option holds, and in particular $x' \not\lesseqgtr_{-j} 0$, $j = \pm 1$. It follows that $x \lesseqgtr_j 0$, $j = \pm 1$, as claimed. □

Lemma 5.3. *Let x, y be impartial games. Then xy is an impartial game and $xy \neq 0 \iff x \neq 0 \wedge y \neq 0$.*

In view of Lemma 5.2 and Lemma 1.14 this shows that multiplication is well-defined modulo $=$ on impartial games.

Proof. We induct on the pair (x, y). The options of xy are

$$(xy)' \equiv x'y + xy' - x'y',$$

and, since the summands are impartial by the inductive hypothesis, this shows that xy is impartial.

Suppose first $x \neq 0, y \neq 0$. Then there exist options $x' = y' = 0$. By the inductive hypothesis we obtain an option

$$(xy)' \equiv x'y + xy' - x'y' = 0,$$

and it follows that $xy \neq 0$.

Suppose now $x = 0$. Then by the inductive hypothesis and Lemma 1.14 every option of xy has the form

$$(xy)' \equiv x'y + xy' - x'y' = x'(y - y').$$

We have to show that this is $\neq 0$ in order to conclude $xy = 0$. To this end note that there is an option $x'' = 0$ and an option $(y - y') \equiv y' - y'$. It follows that the impartial game $x'(y - y')$ has the option

$$x''(y - y') + x'(y' - y') - x''(y' - y').$$

By Lemma 1.14 this is

$$= x''y - x''y' + x'y' - x'y' - x''y' - x''y' = x''y - x''y',$$

and this is $= 0$ by the inductive hypothesis. Hence $x'(y - y') \neq 0$ as claimed.

The case $y = 0$ is analogous to $x = 0$. □

We see that impartial games form an Integral Domain modulo $=$. A mulitplicative identity is $*1 :\equiv \{0|0\}$ (the identity $y(*1) \equiv y$ can be seen by induction).

Lemma 5.4. *Let $x \neq 0$ be an impartial game. Define an impartial game x^{-1} by*

$$0 \in x^{-1}, \quad (x^{-1})' \in x^{-1}, 0 \neq x' \in x \implies (*1 + (x' - x)(x^{-1})')(x')^{-1} \in x^{-1}.$$

Then $x^{-1}x = *1$.

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Proof. By induction on x.

As in the proof of claim 3 in Lemma 2.10 (using Corollary 1.7 and the fact that impartial games form an Integral Domain) we see that the options of $x^{-1}x$ are $\neq *1$.

Moreover, since $0 \in x^{-1}$, we have $x^{-1} \neq 0$, so $x^{-1}x \neq 0$. It follows that some option of $x^{-1}x$ is $= 0$. Applying Lemma 1.11 with $z = *1$ we obtain the claim. □

Hence impartial games modulo $=$ form a Field (of characteristic 2).

Recall that the Sprague–Grundy theorem (see e.g. [Sie13]) states that every finite impartial game is equal to a nimber. This is proved using the mex rule and the substitution rule. Note that the mex rule follows immediately from Lemma 1.11 and the substitution rule from Lemma 1.12.

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