The endpoint multilinear Kakeya theorem via the Borsuk–Ulam/Lusternik–Schnirelmann theorem

After L. Guth, A. Carbery, I. Valdimarsson

Pavel Zorin-Kranich

University of Bonn

Kopp, October 2017

Main theorem

Definition

A 1-*tube* $T \subset \mathbb{R}^n$ is the 1-neighborhood of a straight doubly infinite line in the direction $e(T) \in \mathbb{S}^{n-1}$.
Let $\mathcal{T}_1, \cdots, \mathcal{T}_n$ be families of 1-tubes in $\mathbb{R}^n$ such that $e(T_j)$ is close to the basis vector e_j for $T_j \in \mathcal{T}_j$.

Theorem (Multilinear Kakeya/perturbed Loomis–Whitney)

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathcal{T}_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathcal{T}_n} \chi_{T_n}(x) \right)^{1/(n-1)} dx \lesssim (\#\mathcal{T} \cdots \#\mathcal{T}_n)^{1/(n-1)} .$$

Here and later implicit constants depend only on n.

Main theorem, discrete version

Let $\mathcal{Q}$ denote the lattice of dyadic cubes of unit size. Let also

$$G(Q) = \Big(\prod_j \#\{T_j \in \mathcal{T}_j \mid T_j \cap Q \neq \emptyset\}\Big)^{1/(n-1)}.$$

Then

$$\sum_{Q \in \mathcal{Q}} G(Q) \lesssim \prod_j (\#\mathcal{T}_j)^{1/(n-1)}.$$

Main theorem, discrete version

Let $\mathcal{Q}$ denote the lattice of dyadic cubes of unit size. Let also

$$G(Q) = \left(\prod_j \#\{T_j \in \mathcal{T}_j \mid T_j \cap Q \neq \emptyset\} \right)^{1/(n-1)}.$$

Then

$$\sum_{Q \in \mathcal{Q}} G(Q) \lesssim \prod_j (\#\mathcal{T}_j)^{1/(n-1)}.$$

Equivalent formulation: for every $M : \mathcal{Q} \to \mathbb{R}_+$ with $\sum_Q M(Q) = 1$ there exist $S_j : \mathcal{Q} \to \mathbb{R}_+$ with

$$G(Q) M(Q)^{1/(n-1)} \lesssim \prod_j S_j(Q)^{1/(n-1)},$$

$$\sum_Q S_j(Q) \lesssim \#\mathcal{T}_j.$$

Proof of equivalence (needed direction)

Let $\mathcal{G} := \sum_Q G(Q)$ and $M(Q) = G(Q)/\mathcal{G}$. Then

$$\begin{aligned}
\mathcal{G} &= \Big(\mathcal{G}^{-1/n} \sum_Q G(Q)\Big)^{n/(n-1)} \\
&= \Big(\sum_Q G(Q)^{(n-1)/n} M(Q)^{1/n}\Big)^{n/(n-1)} \\
&\lesssim \Big(\sum_Q \prod_{j=1}^n S_j(Q)^{1/n}\Big)^{n/(n-1)} && \text{(hypothesis)} \\
&\leq \prod_{j=1}^n \big(\sum_Q S_j(Q)\big)^{1/(n-1)} && \text{(Hölder)} \\
&\lesssim \prod_{j=1}^n (\#\mathcal{T}_j)^{1/(n-1)}. && \text{(hypothesis)}
\end{aligned}$$

Not much happened, but cross-interaction and self-interaction are separated.

Ansatz for tubes

$$S_j(Q) = \sum_{T \in \mathcal{T}_j} S_j(Q, T_j)$$

Theorem

For every function $M : \mathcal{Q} \to \mathbb{R}_+$ *with* $\sum M = 1$ *there exist* $S_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}_+$ *with*

$$M(Q) \lesssim \prod_j S_j(Q, T_j) \text{ if } T_j \cap Q \neq \emptyset,$$

$$\sum_{Q \in \mathcal{Q} : T_j \cap Q \neq \emptyset} S_j(Q, T_j) \lesssim 1 \text{ for each } T_j \in \mathcal{T}_j.$$

Wlog M compactly supported.

Ansatz for tubes

$$S_j(Q) = \sum_{T \in \mathcal{T}_j} S_j(Q, T_j)$$

Theorem

For every function $M : \mathcal{Q} \to \mathbb{R}_+$ *with* $\sum M = 1$ *there exist* $S_j : \mathcal{Q} \times \mathcal{T}_j \to \mathbb{R}_+$ *with*

$$M(Q) \lesssim \prod_j S_j(Q, T_j) \text{ if } T_j \cap Q \neq \emptyset,$$

$$\sum_{Q \in \mathcal{Q} : T_j \cap Q \neq \emptyset} S_j(Q, T_j) \lesssim 1 \text{ for each } T_j \in \mathcal{T}_j.$$

Wlog M compactly supported. Will find polynomial p of degree $\lesssim \lambda$ and set

$$S_j(Q, T_j) := \lambda^{-1} \mathfrak{s}_{p,Q}(e(T_j)). \quad \text{Add this to handout!}$$

Directional surface area

Let Z_p be the zero set of the polynomial p. Let

$$\mathfrak{s}_{p,Q}(v) := \int_{Z_p \cap Q} |\langle v, N_x \rangle| \, \mathrm{d}\mathcal{H}^{n-1}(x), \quad N_x \text{ normal unit vector.}$$

(small lie: this is not a continuous function of p, so one has to use a mollified version instead to apply a topological result)

Directional surface area

Let Z_p be the zero set of the polynomial p. Let

$$\mathfrak{s}_{p,Q}(v) := \int_{Z_p \cap Q} |\langle v, N_x \rangle| \, d\mathcal{H}^{n-1}(x), \quad N_x \text{ normal unit vector.}$$

(small lie: this is not a continuous function of p, so one has to use a mollified version instead to apply a topological result)
Guth's tube estimate:

$$\sum_{Q \in \mathcal{Q}, T \cap Q \neq \emptyset} \mathfrak{s}_{p,Q}(e(T)) \leq \int_{Z_p \cap \tilde{T}} |\langle e(T), N_x \rangle| \, d\mathcal{H}^{n-1}(x) \lesssim \deg p.$$

This takes care of the self-interaction term.

Cross-interaction term

Let now $T_j \in \mathcal{T}_j$ be tubes and $Q \in \mathcal{Q}$. Then

$$\prod_j S(Q, T_j) = \lambda^{-n} \prod_j \mathfrak{s}_{p,Q}(e(T_j))$$
$$\sim \lambda^{-n} \Big(\operatorname{vol} \operatorname{conv}(0, e(T_j)/\mathfrak{s}_{p,Q}(e(T_j))) \Big)^{-1}$$

by transversality

$$\gtrsim \lambda^{-n} \Big(\operatorname{vol} \mathbb{B}_{\mathfrak{s}_{p,Q}} \Big)^{-1},$$

where $\mathbb{B}$ is the unit ball of the norm $\mathfrak{s}$.
Want to find polynomial p with

$$\lambda^n M(Q) \lesssim \Big(\operatorname{vol} \mathbb{B}_{\mathfrak{s}_{p,Q}} \Big)^{-1}$$

Visibility

Want to find polynomial p with

$$\tilde{M}(Q) := \lambda^n M(Q) \lesssim \left(\operatorname{vol} \mathbb{B}_{\mathfrak{s}_{p,Q}} \right)^{-1} =: \mathrm{Vis}_{p,Q}\,.$$

Small lie: we pretend that $\mathbb{B}_{\mathfrak{s}_{p,Q}} \subset \mathbb{B}$ for all Q with $M(Q) \neq 0$. For this we need λ to be large enough; this is how we choose λ.
Notice that

$$\sum_{Q} \tilde{M}(Q) = \lambda^n$$

is approximately the dimension of the space of polynomials of degree $\lesssim \lambda$ in n variables. Let $\mathcal{P}^*$ be the unit sphere in this space.

Naive approach to maximizing visibility

Imagine for the moment that the directional surface area is isotropic, i.e. $\mathfrak{s}_{p,Q}(v) \sim s_{p,Q}\|v\|$ for all p, Q, where $s_{p,Q}$ is the usual surface area. In this case:

1. Fit into each cube Q approximately $\tilde{M}(Q)$ disjoint balls of measure $\tilde{M}(Q)^{-1}$.
2. Use the polynomial ham sandwich theorem to find p of degree $\lesssim \lambda$ that bisects all these balls.
3. In each ball Z_p has surface area at least $\tilde{M}(Q)^{-(n-1)/n}$ by the isoperimetric inequality.
4. Summing up gives $s_{p,Q} \gtrsim \tilde{M}(Q)^{1/n}$, hence $\mathbb{B}_{\mathfrak{s}_{p,Q}} \subset \tilde{M}(Q)^{-1/n}\mathbb{B}$, hence $\mathrm{Vis}_{p,Q} \gtrsim \tilde{M}(Q)$.

Problem: $\mathfrak{s}_{p,Q}$ not isotropic.

Numerology of adapted ellipsoids

Suppose now that $\mathbb{B}_{\mathfrak{s}_{p,Q}}$ basically does not depend on p, e.g. in the sense that its John ellipsoid $E_Q \subset \mathbb{B}$ does not depend on p. In this case:

1. We do not have to worry about the cubes with $\operatorname{vol} E_Q \lesssim \tilde{M}(Q)^{-1}$.
2. For the cubes with $\operatorname{vol} E_Q \gg \tilde{M}(Q)^{-1}$ a fixed positive proportion can be covered by $\leq \tilde{M}(Q)$ disjoint copies of ηE_Q for some small absolute constant η.
3. By the polynomial ham sandwich theorem there exists p of degree $\lesssim \lambda$ that bisects all these copies.
4. Let $v_1, \ldots, v_n$ be principal axes of E_Q. In each copy E' of ηE_Q the surface Z_p has area at least $\operatorname{vol}(\eta E_Q)$ in the direction ηv_j for some j (this is an affine invariant formulation of the isoperimetric inequality).
5. Adding these contributions we $\mathfrak{s}_{p,Q}(\eta v_j) \gtrsim 1$ – contradiction for small enough η.

The topological input

Theorem (Lusternik, Schnirelmann, 1930)

If $\mathbb{S}^N$ *is covered by* $N + 1$ *closed sets, then one of these sets contains a pair of antipodal points.*

The topological input

Theorem (Lusternik, Schnirelmann, 1930)

If $\mathbb{S}^N$ is covered by $N+1$ closed sets, then one of these sets contains a pair of antipodal points.

We claim that the sets

$$B(Q) = \{p \in \mathcal{P}^* \mid 1 \leq \mathrm{Vis}_{p,Q} \leq \tilde{M}(Q)\}$$

do not cover $\mathcal{P}^*$. To see this we will write $\cup_Q B(Q)$ as the union of $\lesssim \sum_Q \tilde{M}(Q)$ closed sets that are disjoint from their antipodes.

The topological input

Theorem (Lusternik, Schnirelmann, 1930)

If $\mathbb{S}^N$ is covered by $N+1$ closed sets, then one of these sets contains a pair of antipodal points.

We claim that the sets

$$B(Q) = \{p \in \mathcal{P}^* \mid 1 \leq \mathrm{Vis}_{p,Q} \leq \tilde{M}(Q)\}$$

do not cover $\mathcal{P}^*$. To see this we will write $\cup_Q B(Q)$ as the union of $\lesssim \sum_Q \tilde{M}(Q)$ closed sets that are disjoint from their antipodes. The John ellipsoid $E_{p,Q}$ of $\mathbb{B}_{\mathfrak{s}_{p,Q}}$ depends continuously on p. Using the geometry of the space of all ellipsoids we split the sphere $\mathcal{P}^*$ into $O_n(1)$ symmetric closed subsets $\mathcal{P}^*_\theta$ on each of which $E_{p,Q}$ is locally constant. Let

$$B(Q,\theta) = \{p \in \mathcal{P}^*_\theta \mid 1 \leq \mathrm{Vis}_{p,Q} \leq \tilde{M}(Q)\}.$$

Covering

1. For each ellipsoid $E \subset \mathbb{B}$ with $\operatorname{vol} E \gtrsim \tilde{M}(Q)^{-1}$ fix a maximal collection of disjoint translates of ηE inside each Q, index them by $\alpha = 1, \ldots, C\eta^{-n}\tilde{M}(Q)$. Numerology shows that not each translate of $\eta E_{p,Q}$ can be (approximately) bisected
2. Let

$$B(Q, \theta, \alpha) = \{p \in B(Q, \theta) \text{ not} \approx \text{bisects } \alpha\text{-th copy of } \eta E_{p,Q}\}.$$

   This is a closed set, and it can be partitioned into closed antipodal sets by looking which of $\{x \in Q \mid p(x) > 0\}$ and $\{x \in Q \mid p(x) < 0\}$ is larger.

End of talk

Tanks.

Zhang's extensions to hyperplanes

We can replace tubes T_j by neighborhoods H_j of affine k_j-subspaces (for simplicity with $\sum_{j=1}^{m} k_j = n$). I will not state the results, but will explain the additional ingredients involved in obtaining them.

Let p denote the same polynomial as before and let μ_Q be the pushforward of the surface measure on $Z_p \cap Q$ to $\mathbb{R}^n = \Lambda^1\mathbb{R}^n$ under the normal vector field. In this case we use

$$S_j(Q, H_j) = \lambda^{-k_j} \left| \left\langle H_j, \mu_Q^{\wedge k_j} \right\rangle \right|,$$

where H_j is also used for the volume form on the tangential space of the central affine subpace of H_j. The intersection estimate still holds (but seems to require a fair bit of linear algebra).

Cross-interaction for hyperplanes

Lemma
$|\mu^{\wedge n}| \gtrsim \mathrm{Vis}_{p,Q}$.

Proof.
Note $\mathfrak{s}_{p,Q}(v) = \int |\langle v, w\rangle| \mathrm{d}\mu(w)$. By affine invariance wlog $\mathbb{B}_{\mathfrak{s}_{p,Q}} \sim \mathbb{B}$.
In this case $|\mu| \lesssim 1$ and μ cannot concentracte near hyperplanes. □

Lemma

$$|\mu^{\wedge n}| |\wedge_{j=1}^{m} H_j| \lesssim \prod_{j=1}^{m} |\langle H_j, \mu^{\wedge k_j} \rangle|.$$

Proof.
Laplace expansion formula. □